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The Densities, Correlations and Length Distributions of Vortices Produced at a Gaussian Quench

G. Karra and R. J. Rivers

Blackett Laboratory, Imperial College, London SW7 2BZ

Abstract

We present a model for the formation of relativistic global vortices (strings) at a quench, and calculate their density and correlations. The significance of these results to early universe and condensed-matter physics is discussed. In particular, there is always open, or infinite, string.

1 Introduction

Many systems produce topological defects, in the form of vortices or monopoles, on undergoing a phase transition to an ordered state. In this paper we shall attempt to calculate how such topological debris is produced in simple transitions.

As might have been anticipated, the distributions that we shall find are largely generic, roughly independent both of initial conditions and of the way in which the transition is implemented. It is for this reason that, hitherto, it has not been thought necessary to *derive* defect distributions in any detail. In fact, given our poor understanding of the underlying theories, it has been a consolation to be able to fall back on generic scaling solutions. For example, the large-scale structure of the universe has been attributed [1, 2] to cosmic strings (vortices in the fields) formed at the Grand Unification era. Although we have only a primitive understanding of the relevant field theories, it has been suggested that the initial conditions of any string network are largely washed out after a few expansion times, at which the network is assumed to approach a scaling regime with a few large loops and long strings per horizon volume continuing to produce smaller loops by self and mutual intersection.

Despite this, there are two circumstances in which this lack of detailed initial information leaves us at a loss, to which this paper is largely addressed. The first, of less interest, concerns the *density* of the defects formed. We have known for some time how to make reasonable qualitative estimates[1] of densities and in this paper we can confirm them quantitatively. Secondly, it is not entirely true to say that the details of the string distributions are unimportant. In particular, for the astrophysicists the scaling solutions for the early universe mentioned above arguably require some open or *'infinite'* string, *i.e.*, vortices that do not self-intersect. The presence of such string is, in part, determined by the initial conditions and it is important to know whether it is present for reasonable models and if so, how much. For example, it has been suggested that vortices produced by bubble nucleation in a strong first-order transition will *only* form small loops[3].

The paper is organised as follows. We begin by reiterating the main tactics for determining defect densities and distributions and then display the forms that they will take in a Gaussian approximation for the underlying field distributions. The main tools are the correlation functions of the defect densities. As it stands, some work is necessary to convert them into easily measurable, or easily identifiable, quantities. We present examples, motivated both by our dynamical model (but simpler) and by current numerical simulations in astrophysics, to help us understand them better.

In our model these correlation functions are given a concrete realisation in terms of exponentially growing unstable modes when the phase transition is implemented by an *instantaneous* quench. Although this is an unrealisable idealisation, these results are used as a benchmark for more general transitions when, later, we vary both the initial conditions and the way in which the quench is implemented. It was already implicit in our earlier work[4], of which this is a continuation, that for a very fast quench the initial conditions will, in general, only determine the subleading behaviour of the

defect production. We shall extend the work presented there to look for exceptions to this general behaviour. Further, it will be seen that changing the *rate* at which the quench is implemented can be approximately equivalent to changing the *time* at which the transition can be said to have begun. As a result, the preliminary work of [4] for instantaneous quenches can be extended with only minor modification.

After a discussion of the way in which defects freeze into the field we conclude with some observations about the length distribution of vortices, obtained by interpreting some recent numerical simulations[5, 6] in the light of our model. It will be seen that, whatever we do, there will be infinite string. Yet again we assume flat spacetime, for simplicity. Our conclusions are thus, in this regard, more applicable to weak coupling condensed matter physics than the early universe. Interestingly, in a condensed matter context, the same correlation functions should enable us to estimate the superflow that would occur at a superfluid quench from fluctuations alone[7]. We shall make some steps in this direction.

2 Defect Distributions

Before going into details, some generalities about defect production will be useful. Our main interest is in vortices and our discussion will be centred about them. As will be seen, other defects are similar but simpler. The mechanism for vortex formation (termed the Kibble mechanism[1] in astroparticle physics) is well understood at a qualitative level. In this paper we shall only consider the simplest theory that permits vortices, that of a complex scalar field $\phi(\mathbf{x}, t)$. The complex order parameter of the theory is $\langle\phi\rangle = \eta e^{i\alpha}$ and the theory possesses a global $O(2)$ symmetry that we take to be broken at its phase transition. Initially, we take the system to be in the symmetry-unbroken (disordered) phase, in which the field is distributed about $\phi = 0$ with zero mean. We assume that, at some time $t = t_0$, the $O(2)$ symmetry of the ground-state (vacuum) is broken by a rapid change in the environment inducing an explicit time-dependence in the field parameters. Once this quench is completed the ϕ-field potential takes the familiar symmetry-broken form $V(\phi) = -M^2|\phi^2| + \lambda|\phi^4|/4$ with $M^2 > 0$.

The transition for such a global symmetry is *continuous* and we expect that, as the complex scalar field begins to fall from the false ground-state into the true ground-state, different points on the ground-state manifold (the circle S^1, labelled by the phase α of $\langle\phi\rangle$) will be chosen at each point in space [1]. If this is so then continuity and single valuedness will sometimes force the field to remain in the false ground-state at $\phi = 0$. For example, the phase of the field may change by an integer multiple of 2π on going round a loop in space. This requires at least one *zero* of the field within the loop, each of which has topological stability and characterises a vortex (or string). As to the density of the strings, if the phase α is correlated over a distance ξ, then the density of strings

[1] Had it been a first-order transition the field would have tunnelled towards the groundstate, with very different consequences for densities and correlations than those described here

passing through any surface will be $O(\xi^{-2})$ *i.e.* a fraction of a string per unit correlation area. The assumed lack of correlation of the field phase over larger distances than ξ (inevitable on causal grounds in cosmological models) is interpreted as saying that the power $P(k)$ of the field fluctuations, defined by

$$\langle\tilde{\phi}(\mathbf{k})\tilde{\phi}^*(\mathbf{k}')\rangle = (2\pi)^3 P(k)\delta^{(3)}(\mathbf{k}-\mathbf{k}') \tag{2.1}$$

has the form $P(k) = O(k^0)$ for small k. That is, $P(k)$ describes *white noise* at large distances. Several numerical simulations based on this assumption have been performed. The standard simulation, by Vachasparti and Vilenkin[8], assumes a regular cubic lattice, in whose cells the $O(2)$ field phase is chosen at random. This assumption of 'white noise' leads to approximately 80% of the string network being in open string. While this number is lattice dependent[9] there is no doubt that a substantial fraction of string is open, or 'infinite'. We shall take these white noise field fluctuation predictions as a further benchmark against which to compare our dynamical predictions. On the completion of the transition, when fluctuations are too weak to eliminate or create strings, a network of strings survives whose further evolution is determined by classical considerations as the field gradients adjust to minimise the energy.

Vortices are not the only defects permitted by global symmetries although, ultimately, they are the only ones of real interest to us. More generally, a global $O(N)$ scalar theory permits defects with integer topological charge in $D = N$ spatial dimensions (monopoles) or $D = N + 1$ spatial dimensions (vortices). Because of their relative complexity it is helpful to try out our methods on monopoles, which include the *kinks* on the line for a real scalar field theory ($N = D = 1$) as a special case, and the *monopoles* in the plane for a complex scalar $O(2)$ theory ($N = D = 2$). [The case of $O(3)$ global monopoles in three-dimensional space was examined by us elsewhere [4] and we shall not consider it further]. With minor qualifications[2] a similar mechanism of continuous phase (or field) separation as that for vortex production is equally valid for the formation of these defects also.

The question is, how can we infer these vortex, and other defect, densities and the density correlations from the microscopic field dynamics? The answer lies in the fact noted earlier, that for the global $O(2)$ theory of a complex scalar field $\phi = (\phi_1 + \phi_2)/\sqrt{2}$, ($\phi_1, \phi_2$ real) the string core is a line of zeroes of the fields ϕ_a (a=1,2). The characterisation of an $O(N)$ global defect by its field zeroes is equally true for vortices in the plane ($N = 2$) and kinks on the line ($N = 1$).

The problem is solved if we can identify those zeroes which will freeze out to define the late-time defects. This will require careful winnowing, since it is apparent that quantum fluctuations lead to zeroes of the fields on all distance scales (even in the disordered phase). For the moment we ignore this difficulty, and attempt to count *every* zero. The problem then reduces to that of determining the distribution of field zeroes, given the

[2]Because of the peculiarities of one spatial dimension, that would confuse the issue, we pretend that we are examining a one-dimensional section of a real field in higher dimensions and ignoring other degrees of freedom.

distribution of fields. This has been discussed in the literature on several occasions. We shall call repeatedly on the work of Halperin [10] and Mazenko and Liu [11], but see also Bray [12].

2.1 Kinks on the line

As a prologue to the more difficult problem of vortices in three dimensions, we begin with the much simpler problem of identifying the zeroes of a real field $\phi(x,t)$ in *one* [3] space dimension. To see how to proceed, consider an ensemble of systems evolving from one of a set of disordered states whose relative probabilities are known, to an ordered state as indicated above. At any given time t, the field will adopt one of the possible configurations $\Phi(x)$, whose zeroes we wish to track. As the field evolves to its equilibrium values (one of the two minima of its potential $V(\phi) = -M^2\phi^2 + \lambda\phi^4/4$) these zeroes fluctuate and annihilate, but some of them will come to define the positions x of 'kinks' (field interpolations from one minimum to the other at which $\Phi'(x) > 0$), some the position of 'antikinks' (at which $\Phi'(x) < 0$), the one-dimensional counterparts of vortices and 'anti'-vortices.

Suppose, at a given time, the zeroes of $\Phi(x)$ occur at $x = x_1, x_2, ...$ It is useful to define *two* densities. The first,

$$\bar{\rho}(x) = \sum_i \delta(x - x_i), \tag{2.2}$$

is the *total* density of zeroes, not distinguishing between kink zeroes and antikink zeroes (by which we now mean zeroes at which the field has positive or negative derivative). The second is the *topological* density,

$$\rho(x) = \sum_i n_i \delta(x - x_i), \tag{2.3}$$

where $n_i = \mathrm{sign}(\Phi'(x_i))$, measuring (net) topological charge, the number of kink minus the number of antikink zeroes.

Equivalently, in terms of the Φ-field, the total density is

$$\bar{\rho}(x) = \delta[\Phi(x)]|\Phi'(x)|, \tag{2.4}$$

since $\Phi'(x)$ is the Jacobian of the transformation from zeroes to fields. Similarly, the topological density is

$$\rho(x) = \delta[\Phi(x)]\Phi'(x). \tag{2.5}$$

Analytically, it is not possible to keep track of individual transitions, but we can construct ensemble averages. If the phase change begins at time t_0 then, for $t > t_0$, it is possible in principle to calculate the probability $p_t[\Phi]$ that $\phi(x,t)$ takes the value $\Phi(x)$

[3] See earlier footnote.

at time t. Ensemble averaging $\langle F[\Phi]\rangle_t$ at time t is understood as averaging over the field probabilities $p_t[\Phi]$. This is not thermal averaging since we are out of equilibrium.

The situation we have in mind is one in which, for early times after the transition when the available space permits many domains,

$$\langle\rho(x)\rangle_t = 0, \tag{2.6}$$

*i.e.*an equal likelihood of a kink zero or an antikink zero occurring in an infinitesimal length, compatible with an initially disordered state. However, the total zero density

$$\begin{aligned} \bar{n}(t) &= \langle\bar{\rho}(x)\rangle_t \\ &= \int \mathcal{D}\Phi \; p_t[\Phi]\delta[\Phi(x)]|\Phi'(x)| \; > \; 0 \end{aligned} \tag{2.7}$$

is positive. The distribution of the zeroes is given by the density correlation function

$$\begin{aligned} C(x;t) &= \langle\rho(x)\rho(0)\rangle_t \\ &= \int \mathcal{D}\Phi \; p_t[\Phi]\delta[\Phi(x)]\delta[\Phi(0)]\Phi'(x)\Phi'(0), \end{aligned} \tag{2.8}$$

$(x \neq 0)$ which will also be non-zero.

To make this relationship more concrete, we observe that, on separating out the diagonal and non-diagonal terms in the expansion for $\rho(x)\rho(y)$ from (2.3), then

$$\rho(x)\rho(y) = \bar{\rho}(x)\delta(x-y) + g(x-y) \tag{2.9}$$

where

$$g(x-y) = \sum_{i\neq j} n_i n_j \delta(x-x_i)\delta(y-x_j). \tag{2.10}$$

That is,

$$\langle\rho(x)\rho(0)\rangle_t = \bar{n}(t)\delta(x) + C(x;t). \tag{2.11}$$

where $C(x;t) = \langle g(x)\rangle_t$. Charge conservation

$$\int_{-\infty}^{\infty} dx \, \langle\rho(x)\rho(0)\rangle_t = 0 \tag{2.12}$$

implies

$$\int_{-\infty}^{\infty} dx \, C(x;t) = -\bar{n}(t), \tag{2.13}$$

requiring that $C(x;t) = C(-x;t)$ be largely negative.

We can now relate $C(x;t)$ to the distribution and spacing of zeroes by calculating the variance of the topological charge

$$n_L = \int_0^L dx \, \rho(x) \tag{2.14}$$

on the interval $I = [0, L]$. In this particularly simple case $n_L = -1, 0, 1$. Then

$$\begin{aligned}(\Delta_t n_L)^2 &= \langle n_L^2\rangle_t \\ &= \int_0^L dx \int_0^L dy \, \langle\rho(x)\rho(y)\rangle_t \\ &= L\bar{n}(t) + \int_0^L dx \int_0^L dy \, C(x-y;t) \\ &= -\int_{x<0}^{x>L} dx \int_0^L dy \, C(x-y;t).\end{aligned} \tag{2.15}$$

from (2.13).

If $h(x;t)$ is defined by $C(x;t) = \partial h(x;t)/\partial x$ then

$$\frac{\partial(\Delta_t n_L)^2}{\partial L} = -2h(L;t). \tag{2.16}$$

However, if p_L is the probability that, on average, a length L of the line contains an odd number of zeroes, then $(\Delta_t n_L)^2 = p_L$. Thus

$$C(L;t) = h'(L;t) = -\frac{1}{2}\frac{d^2 p_L}{dL^2}. \tag{2.17}$$

As an extreme case we note that, for an array of equally spaced zeroes, separation $\xi(t)$, p_L is saw-toothed, period 2ξ, from which

$$h(x;t) = \frac{1}{2\pi\xi(t)}\mathrm{sign}(\sin(\pi x/\xi(t))), \tag{2.18}$$

whence $C(x;t) = h'(x;t)$ is a sum of δ-functions. At the other extreme, independent (Poisson) zeroes, mean separation $\xi(t)$, give

$$h(x;t) = e^{-2|x|/\xi(t)}/2\xi(t) \tag{2.19}$$

and

$$C(L;t) = -e^{-2L/\xi(t)}/\xi^2(t). \tag{2.20}$$

These provide a useful guide, and we shall return to them later. However, some caution is necessary since, on the line, kink zero is followed by anti-kink zero and vice-versa. As an intermediate step between this and the vortices of the complex field $\phi(\mathbf{x}, t)$ in three dimensions, we shall consider the same field in two dimensions, for which monopoles (point defects) occur.

2.2 Monopoles in two dimensions

An $O(2)$ complex field $\Phi(\mathbf{x})$ in the plane permits monopoles, at this stage identified by zeroes of the field with non-trivial winding number. After the transition is completed,

the relevant zeroes will define the centres of effectively classical monopoles. Although Derrick's theorem prohibits finite-energy static monopole solutions to the classical field equations, a non-zero monopole density provides a cut-off to the logarithmic tails of the individual monopoles, and there is no problem on this score.

For the moment we continue to count every field zero, at whatever scale and however transient. If $\Phi(\mathbf{x})$ has zeroes at $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, ...$ the total and topological densities $\bar{\rho}(\mathbf{x})$ and $\rho(\mathbf{x})$ are the straightforward generalisations of (2.2) and (2.3),

$$\bar{\rho}(\mathbf{x}) = \sum_i \delta(\mathbf{x} - \mathbf{x}_i), \tag{2.21}$$

and

$$\rho(\mathbf{x}) = \sum_i n_i \delta(\mathbf{x} - \mathbf{x}_i), \tag{2.22}$$

where $n_i = \pm 1$ is the winding number of the zero (higher winding numbers are taken as multiple zeroes).

The relationship between zeroes and fields is through the Jacobian, giving

$$\rho(\mathbf{x}) = \delta^2[\Phi(\mathbf{x})]\epsilon_{jk}\partial_j\Phi_1(\mathbf{x})\partial_k\Phi_2(\mathbf{x}), \;\; i,j = 1,2 \tag{2.23}$$

where $\epsilon_{12} = -\epsilon_{21} = 1$, otherwise zero. As before we assume that it is possible in principle to calculate the probability $p_t[\Phi]$ that $\phi(\mathbf{x}, t)$ takes the value $\Phi(\mathbf{x})$ at time t [4], from which we define the ensemble averages $\langle F[\Phi]\rangle_t$. The topological charge average $\langle\rho(\mathbf{x})\rangle_t$ is taken to be zero as before. The non-zero total density and the topological density correlation functions are defined as for the one-dimensional case,

$$\begin{aligned} \bar{n}(t) &= \langle\bar{\rho}(\mathbf{x})\rangle_t \\ &= \int \mathcal{D}\Phi \; p_t[\Phi]\delta^2[\Phi(\mathbf{x})]|\epsilon_{jk}\partial_j\Phi_1(\mathbf{x})\partial_k\Phi_2(\mathbf{x})| \; > 0 \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} C(r;t) &= \langle\rho(\mathbf{x})\rho(\mathbf{0})\rangle_t \\ &= \int \mathcal{D}\Phi \; p_t[\Phi]\delta^2[\Phi(\mathbf{x})]\delta^2[\Phi(\mathbf{0})]\epsilon_{jk}\partial_j\Phi_1(\mathbf{x})\partial_k\Phi_2(\mathbf{x})\epsilon_{lm}\partial_l\Phi_1(\mathbf{0})\partial_m\Phi_2(\mathbf{0}) \end{aligned} \tag{2.25}$$

$(r = |\mathbf{x}| \neq \mathbf{0})$. Charge conservation again applies, as

$$\int d^2x \, C(r;t) = -\bar{n}(t), \tag{2.26}$$

but it is not so easy to give a direct interpretation to $C(\mathbf{x};t)$ in terms of monopole separations.

[4]Throughout, it will be convenient to decompose Φ into real and imaginary parts as $\Phi = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2)$. This is because we wish to track the field as it falls from the unstable ground-state hump at the centre of the potential to the ground-state manifold in Cartesian field space.

Nonetheless, it is still useful to consider the variance in the topological charge

$$n_S = \int_{\mathbf{x}\in S} d^2x\, \rho(\mathbf{x}), \tag{2.27}$$

this time through a region S in the plane, (area s) since it can be an observable quantity. For example, on quenching ^4He in an annulus S, $(\Delta_t n_S)^2$ is a measure of the supercurrent generated by the quench [7]. Nothing that we have said so far really requires a relativistic theory, so let us pursue it a little further. Then, from (2.26) it follows that

$$\begin{aligned}(\Delta_t n_S)^2 &= \int_{\mathbf{x}\in S} d^2x \int_{\mathbf{y}\in S} d^2y\, \langle\rho(\mathbf{x})\rho(\mathbf{y})\rangle_t \\ &= -\int_{\mathbf{x}\notin S} d^2x \int_{\mathbf{y}\in S} d^2y\; C(|\mathbf{x}-\mathbf{y}|;t) \end{aligned}\tag{2.28}$$

the two-dimensional counterpart to (2.15). Suppose that there are short-range vortex-zero antivortex-zero correlations in which $C(r;t)$ is short-range in r, with lengthscale $\xi(t)$. If $\xi(t)$ is the only lengthscale then $C(r;t)$ is $O(\xi^{-4}(t))$.

With $\mathbf{x}$ outside S, and $\mathbf{y}$ inside S, all the contribution to $(\Delta_t n_S)^2$ comes from the vicinity of the boundary of S, rather than the whole area. More precisely, suppose that S is a disc of radius L [5]. Then (2.28) can be written as

$$(\Delta_t n_S)^2 = -2\pi \int_0^L r\,dr \int_L^\infty r'\,dr' \int_0^{2\pi} d\theta\; C(R;t) \tag{2.29}$$

where

$$R^2 = r^2 + r'^2 - 2rr'\cos\theta. \tag{2.30}$$

The integration region in r, r' is restricted to $|r'-L| = O(\xi(t)) = |L-r|$. If $C(r;t)$ is non-singular at $r=0$ and not varying too rapidly at the origin, then the θ-integration is limited to a range $O(\xi(t)/L)$, whence

$$(\Delta_t n_S)^2 = O\Big(\frac{L}{\xi(t)}\Big). \tag{2.31}$$

This robust result is compatible with, but does not imply, a random walk in field phase around the perimeter and we shall return to it later, when we shall see that our model implies short-range correlations [6].

2.3 Vortices in Three Dimensions

Finally reaching our goal of vortices in three dimensions, we define the *topological line density* of zeroes $\boldsymbol{\rho}(\mathbf{r})$ [10, 11] by

$$\boldsymbol{\rho}(\mathbf{x}) = \sum_n \int ds \frac{d\mathbf{R}_n}{ds}\delta^3[\mathbf{x}-\mathbf{R}_n(s)]. \tag{2.32}$$

[5] The extension to an annulus is straightforward.

[6] At the other extreme we note that, if the monopoles and antimonopoles are individually and mutually uncorrelated (a double Poisson distribution), then $(\Delta n_S)^2 = s\bar{n}$. That is, the variance grows linearly with area.

In (2.1) ds is the incremental length along the line of zeroes $\mathbf{R}_n(s)$ (n=1,2,.. .) and $\frac{d\mathbf{R}_n}{ds}$ is a unit vector pointing in the direction which corresponds to positive winding number. As in the previous case, we begin by counting all zeroes, anticipating that the relevant ones will become the cores of the strings of the resulting network. Yet again, a finite string density once the transition is complete provides the cutoff necessary to eliminate the infrared divergent tails of the energy densities that we would expect from Derrick's theorem.

It follows that, in terms of the zeroes of $\Phi(\mathbf{x})$, $\rho_i(\mathbf{x})$ can be written as

$$\rho_i(\mathbf{x}) = \delta^2[\Phi(\mathbf{x})]\epsilon_{ijk}\partial_j\Phi_1(\mathbf{x})\partial_k\Phi_2(\mathbf{x}), \tag{2.33}$$

where $\delta^2[\Phi(\mathbf{x})] = \delta[\Phi_1(\mathbf{x})]\delta[\Phi_2(\mathbf{x})]$. The coefficient of the δ-function in (2.3) is the Jacobian of the more complicated transformation from line zeroes to field zeroes. We shall also need the *total line density* $\bar{\rho}(\mathbf{x})$, the counterpart of $\bar{\rho}(x)$ of (2.4),

$$\bar{\rho}_i(\mathbf{x}) = \delta^2[\Phi(\mathbf{x})]|\epsilon_{ijk}\partial_j\Phi_1(\mathbf{x})\partial_k\Phi_2(\mathbf{x})|. \tag{2.34}$$

As before, ensemble averaging $\langle F[\Phi]\rangle_t$ at time t means averaging over the field probabilities $p_t[\Phi]$. Again we assume

$$\langle\rho_j(\mathbf{x})\rangle_t = 0. \tag{2.35}$$

i.e. an equal likelihood of a string line-zero or an antistring line-zero passing through an infinitesimal area. However,

$$\bar{n}(t) = \ \langle\bar{\rho}_i(\mathbf{x})\rangle_t \ > 0 \tag{2.36}$$

and measures the *total* line-zero density in the direction i, without regard to string orientation. The isotropy of the initial state guarantees that $\bar{n}(t)$ is independent of the direction i. Further, the line density correlation functions

$$C_{ij}(\mathbf{x};t) = \langle\rho_i(\mathbf{x})\rho_j(\mathbf{0})\rangle_t \tag{2.37}$$

will be non-zero, and give information on the persistence length of line zeroes. It will be convenient, for later work, to decompose $C_{ij}(\mathbf{x};t)$ as ($r = |\mathbf{x}|$),

$$C_{ij}(\mathbf{x};t) = A(r;t)\Big(\delta_{ij} - \frac{x_ix_j}{r^2}\Big) + B(r;t)\Big(\frac{x_ix_j}{r^2}\Big), \tag{2.38}$$

The realisations of $\bar{n}(t)$ and $C_{ij}(\mathbf{x};t)$ in terms of $p_t[\Phi]$ are simple generalisations of (2.24) and (2.25) and will not be given explicitly.

Charge conservation now means

$$\int d^3x\ C_{ij}(\mathbf{x};t) = 0 \tag{2.39}$$

without any inhomogeneous term. On taking the trace in (2.39) it follows that

$$\int d^3x\ \Big(2A(r;t) + B(r;t)\Big) = 0 \tag{2.40}$$

We note that, from (2.32), the integral of $\rho_j(\mathbf{x})$ over an open surface S with normal in the j-direction does not measure the winding number along the boundary of S since each line zero is weighted by the cosine of its angle of incidence on S. Thus the variance of the winding number (topological charge) through S is essentially the two-dimensional problem discussed previously.

There are no simple quantities that can be calculated for line zeroes (e.g.Poisson strings). However, we note that, if $\mathbf{x} = (0, 0, r)$, then

$$C_{11}(\mathbf{x}; t) = C_{22}(\mathbf{x}; t) = A(r; t) \tag{2.41}$$

measures the likelihood of there being a string zero (or antistring zero) in the same direction at separation r. A negative value of A of $O(\bar{n}^2)$ indicates the presence of vortex-antivortex line zero pairs at separation r. Further, for the same r,

$$C_{33}(\mathbf{x}; t) = B(r; t) \tag{2.42}$$

measures the tendency of a vortex to bend on a distance r (provided r is sufficiently small for it to be measuring the same string). Once $B\bar{n}^2$ is negligable the line has bent away from its initial direction.

3 Ensemble Averaging

In relating distributions of defects to field fluctuations we have seen that we need the field probability $p_t[\Phi]$ at all times. There is no difficulty in writing a formal expression for $p_t[\Phi]$, although its calculation is another matter. Details are given in [4], and we shall only provide the briefest recapitulation.

Take $t = t_0$ as our starting time. Suppose that, at t_0, the system is in a pure state, in which the measurement of ϕ would give $\Phi_0(\mathbf{x})$. That is:-

$$\hat{\phi}(t_0, \mathbf{x})|\Phi_0, t_0\rangle = \Phi_0|\Phi_0, t_0\rangle. \tag{3.43}$$

The probability $p_{t_f}[\Phi_f]$ that, at time $t_f > t_0$, the measurement of ϕ will give the value Φ_f is $p_{t_f}[\Phi_f] = |\Psi_{f0}|^2$, where Ψ_{f0} is the state-functional with the specified initial condition. As a path integral

$$\Psi_{f0} = \int_{\phi(t_0)=\Phi_0}^{\phi(t_f)=\Phi_f} \mathcal{D}\phi \, \exp\Big\{iS_t[\phi]\Big\}, \tag{3.44}$$

where $S_t[\phi]$ is the (time-dependent) action that desribes how the field ϕ is driven by the environment, $\mathcal{D}\phi = \prod_{a=1}^{N} \mathcal{D}\phi_a$ and spatial labels have been suppressed. Specifically, for $t > t_0$ the action for the field is taken to be

$$S_t[\phi] = \int d^{D+1}x \Big(\frac{1}{2}\partial_\mu\phi_a\partial^\mu\phi_a - \frac{1}{2}m^2(t)\phi_a^2 - \frac{1}{4}\lambda(t)(\phi_a^2)^2\Big). \tag{3.45}$$

where $m(t)$, $\lambda(t)$ describe the evolution of the parameters of the theory under external influences, to which the field responds. The spatial dimension $D = 3$ for the relevant

case of vortices, and $D = 2$ for monopoles in the plane [7]. As with Φ, it is convenient to decompose the complex field ϕ in terms of two massive real scalar fields ϕ_a, $a = 1, 2$ as $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, in terms of which $S[\phi]$ shows a global $O(2)$ invariance, broken by the mass term if $m^2(t)$ is negative.

It follows that $p_{t_f}[\Phi_f]$ can be written in the closed time-path form

$$p_{t_f}[\Phi_f] = \int_{\phi_\pm(t_0)=\Phi_0}^{\phi_\pm(t_f)=\Phi_f} \mathcal{D}\phi_+ \mathcal{D}\phi_- \, \exp\Big\{ i\Big(S_t[\phi_+] - S_t[\phi_-] \Big) \Big\}. \tag{3.46}$$

Instead of separately integrating $\phi_\pm$ along the time paths $t_0 \le t \le t_f$, the integral can be interpreted as time-ordering of a field ϕ along the closed path $C_+ \oplus C_-$ where $\phi = \phi_+$ on C_+ and $\phi = \phi_-$ on C_-. The two-field notation is misleading in that it suggests that the ϕ_+ and ϕ_- fields are decoupled. That this is not so follows immediately from the fact that $\phi_+(t_f) = \phi_-(t_f)$. It is necessary to keep this in mind when we extend the contour from t_f to $t = \infty$. Either ϕ_+ or ϕ_- is an equally good candidate for the physical field, but we choose ϕ_+. With this choice and suitable normalisation, p_{t_f} becomes

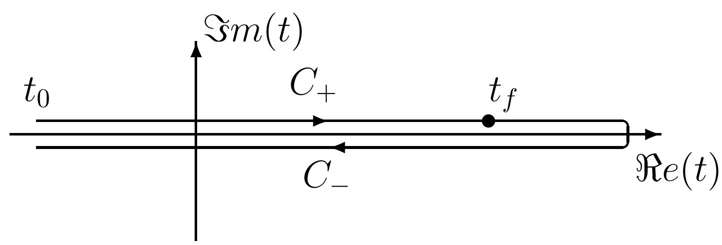

Figure 1: The closed timepath contour $C_+ \oplus C_-$.

$$p_{t_f}[\Phi_f] = \int_{\phi_\pm(t_0)=\Phi_0} \mathcal{D}\phi_+ \mathcal{D}\phi_- \, \delta[\phi_+(t) - \Phi_f] \, \exp\Big\{ i\Big(S_t[\phi_+] - S_t[\phi_-] \Big) \Big\}, \tag{3.47}$$

where $\delta[\phi_+(t) - \Phi_f]$ is a delta functional, imposing the constraint $\phi_+(t, \mathbf{x}) = \Phi_f(\mathbf{x})$ for each $\mathbf{x}$.

The choice of a pure state at time t_0 is too simple to be of any use. The one fixed condition is that we begin in a symmetric state with $\langle\phi\rangle = 0$ at time $t = t_0$. Otherwise, our ignorance is parametrised in the probability distribution that at time t_0, $\phi(t_0, \mathbf{x}) = \Phi(\mathbf{x})$. If we allow for an initial probability distribution $p_{t_0}[\Phi]$ then $p_{t_f}[\Phi_f]$ is generalised to

$$p_{t_f}[\Phi_f] = \int \mathcal{D}\Phi p_{t_0}[\Phi] \int_{\phi_\pm(t_0)=\Phi} \mathcal{D}\phi_+ \mathcal{D}\phi_- \, \delta[\phi_+(t_f) - \Phi_f] \, \exp\Big\{ i\Big(S_t[\phi_+] - S_t[\phi_-] \Big) \Big\}. \tag{3.48}$$

[7] $D = 1$ for kinks on the line

It is impossible to derive p_t analytically for general initial conditions. Fortunately, we shall see that, in many circumstances, the details of the initial condition are largely irrelevant. All the cases that we might wish to consider are encompassed in the assumption that Φ is Boltzmann distributed at time t_0 at an effective temperature of $T_0 = \beta_0^{-1}$ according to a Hamiltonian $H_0[\Phi]$, where the subscript *zero* denotes t_0 rather than a free field. That is

$$p_{t_0}[\Phi] = \langle \Phi, t_0 | e^{-\beta_0 H_0} | \Phi, t_0 \rangle = \int_{\phi_3(t_0)=\Phi=\phi_3(t_0-i\beta_0)} \mathcal{D}\phi_3 \exp\Big\{ iS_0[\phi_3] \Big\}, \tag{3.49}$$

for a corresponding action $S_0[\phi_3]$, in which ϕ_3 is taken to be periodic in imaginary time with period β_0. We take $S_0[\phi_3]$ to have the standard form in ϕ_3 as

$$S_0[\phi_3] = \int d^{D+1}x \Big[\frac{1}{2}(\partial_\mu \phi_{3\,a})(\partial^\mu \phi_{3\,a}) - \frac{1}{2} m_0^2 \phi_{3\,a}^2 - \frac{1}{4}\lambda_0 (\phi_{3\,a}^2)^2 \Big]. \tag{3.50}$$

We stress that m_0, λ_0 and β_0 parametrise our uncertainty in the initial conditions. The choice $\beta_0 \to \infty$ corresponds to choosing the $p_t[\Phi]$ to be determined by the ground state functional of H_0, for example. For the sake of argument we take $T_0 = \beta_0^{-1}$ to be a temperature higher than the transition temperature T_c and $m_0 = m(T_0)$ $(m_0^2 > 0)$ to be the effective mass at this at this temperature. Whatever, the effect is to give an action $S_3[\phi]$ in which we are in thermal equilibrium for $t < t_0$ during which period the mass $m(t)$ and coupling constant $\lambda(t)$ take the constant values m_0 and λ_0 respectively.

We now have the explicit form for $p_{t_f}[\Phi_f]$,

$$p_{t_f}[\Phi_f] = \int_B \mathcal{D}\phi_3 \mathcal{D}\phi_+ \mathcal{D}\phi_- \exp\Big\{ iS_0[\phi_3] + i(S[\phi_+] - S[\phi_-]) \Big\} \, \delta[\phi_+(t_f) - \Phi_f], \tag{3.51}$$

where the boundary condition B is $\phi_\pm(t_0) = \phi_3(t_0) = \phi_3(t_0 - i\beta_0)$. This can be written as the time ordering of a single field:-

$$p_{t_f}[\Phi_f] = \int_B \mathcal{D}\phi \, e^{iS_C[\phi]} \, \delta[\phi_+(t_f) - \Phi_f], \tag{3.52}$$

along the contour $C = C_+ \oplus C_- \oplus C_3$, extended to include a third imaginary leg, where ϕ takes the values ϕ_+, ϕ_- and ϕ_3 on C_+, C_- and C_3 respectively, for which S_C is $S[\phi_+]$, $S[\phi_-]$ and $S_0[\phi_3]$. Henceforth we drop the suffix f on Φ_f and take the origin in time from which the evolution begins as $t_0 = 0$.

We perform one final manoeuvre with $p_t[\Phi]$ before resorting to further approximation to demonstrate how we can average without having to calculate $p_t[\Phi]$ explicitly. Further, this will enable us to avoid a nominally ill-defined inversion of a two-point function later on, a consequence of the seeming independence of ϕ_+ and ϕ_- mentioned earlier. Consider the generating functional:-

$$Z[j_+, j_-, j_3] = \int_B \mathcal{D}\phi \, \exp\Big\{ iS_C[\phi] + i \int j\phi \Big\}, \tag{3.53}$$

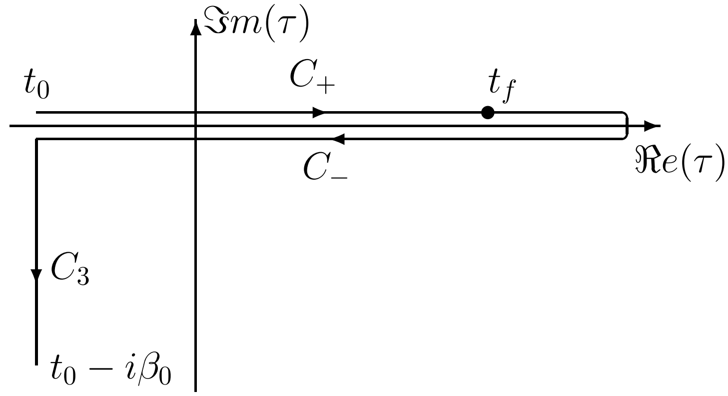

Figure 2: A third imaginary leg

where $\int j\phi$ is a short notation for:-

$$\int j\phi \equiv \int_0^\infty dt\ [\,j_+(t)\phi_+(t) - j_-\phi_-(t)\,] + \int_0^{-i\beta} j_3(t)\phi_3(t)\,dt, \tag{3.54}$$

omitting spatial arguments. Then introducing $\alpha_a(\mathbf{x})$ where $a = 1, \ldots, N$, we find:-

$$\begin{aligned} p_{t_f}[\Phi] &= \int \mathcal{D}\alpha \int_B \mathcal{D}\phi\ \exp\Big\{iS_C[\phi]\Big\}\ \exp\Big\{i\int d^D x\alpha_a(\mathbf{x})[\phi_+(t_f,\mathbf{x}) - \Phi(\mathbf{x})]_a\Big\} \\ &= \int \mathcal{D}\alpha\ \exp\Big\{-i\int \alpha_a\Phi_a\Big\}\, Z[\overline{\alpha}, 0, 0], \end{aligned}$$

where $\overline{\alpha}$ is the source $\overline{\alpha}(t,\mathbf{x}) = \alpha(\mathbf{x})\delta(t - t_f)$. As with $\mathcal{D}\phi$, $\mathcal{D}\alpha$ denotes $\prod_1^N \mathcal{D}\alpha_a$. Ensemble averages are now expressible in terms of Z_μ. Of particular relevance, $W_{ab}(|\mathbf{x}-\mathbf{x}'|;t) = \langle\Phi_a(\mathbf{x})\Phi_b(\mathbf{x}')\rangle_t$ is given by

$$\begin{aligned} W_{ab}(|\mathbf{x}-\mathbf{x}'|;t) &= \int \mathcal{D}\Phi\ \Phi_a(\mathbf{x})\Phi_b(\mathbf{x}') \int \mathcal{D}\alpha\ \exp\Big\{-i\int \alpha_a\Phi_a\Big\}\ Z_\mu[\overline{\alpha}, 0, 0] \\ &= -\int \mathcal{D}\alpha\ \frac{\delta^2}{\delta\alpha_a(\mathbf{x})\delta\alpha_b(\mathbf{x}')} \int \mathcal{D}\Phi\ \exp\Big\{-i\int \alpha_a\Phi_a\Big\}\ Z_\mu[\overline{\alpha}, 0, 0] \\ &= -\int \mathcal{D}\alpha\ \frac{\delta^2}{\delta\alpha_a(\mathbf{x})\delta\alpha_b(\mathbf{x}')} \Big\{\delta^2\,[\alpha]\ Z_\mu[\overline{\alpha}, 0, 0]\Big\} \end{aligned} \tag{3.55}$$

On integrating by parts

$$\begin{aligned} W_{ab}(|\mathbf{x}-\mathbf{x}'|;t) &= -\frac{\delta^2}{\delta\alpha_a(\mathbf{x})\delta\alpha_b(\mathbf{x}')} Z_\mu[\overline{\alpha}, 0, 0]\Big|_{\alpha=0} \\ &= \langle\phi_a(\mathbf{x},t)\phi_b(\mathbf{x}',t)\rangle, \end{aligned} \tag{3.56}$$

the equal-time thermal Wightman function with the given thermal boundary conditions. Because of the non-equilibrium time evolution there is no time translation invariance in the double time label.

4 A Gaussian Model for Defect Distributions

We have yet to specify the nature of the quench but it is already apparent that, if we are to make further progress, additional approximations are necessary. We return to our one-dimensional example.

4.1 Kinks

Suppose the fields and their derivatives at the same point show approximate independence. Then the zero density $\bar{n}(t)$ of (2.7) separates as

$$\bar{n}(t) \approx \langle\delta[\Phi(x)]\rangle_t \langle|\Phi'(x)|\rangle_t. \tag{4.57}$$

and we can estimate, or bound, each factor separately. Specifically, on writing the δ-function term as

$$\langle\delta[\Phi(x)]\rangle_t = \int đ\alpha \langle \exp\{i\int dy\, j(y)\Phi(y)\}\rangle_t \tag{4.58}$$

where $j(y) = \alpha\delta(y-x)$, it follows that

$$\bar{n}(t) \leq \frac{1}{\sqrt{2\pi}}\left|\frac{W''(0;t)}{W(0;t)}\right|^{\frac{1}{2}}\Big[1 + \text{Connected Correlation Functions}\Big] \tag{4.59}$$

on using the Schwarz inequality, where primes on W denote differentiation with respect to x. Some care is needed. The connected correlation functions that appear in (4.59) are $O(\lambda)$ but depend on time, growing as the field grows away from $\phi = 0$. As is well understood, perturbation theory breaks down at long enough times. However, for small times and weak coupling the first term in (4.59) is reliable. We see that, if the Fourier transform $\tilde{W}(k;t)$ of $W(x;t)$ is dominated by wave vectors $k(t) = O(\xi^{-1}(t))$ at time t, then

$$\bar{n}(t) \leq O\Big(\frac{1}{\xi(t)}\Big). \tag{4.60}$$

This shows that $\xi(t)$ sets the domain size.

A similar qualitative analysis can be attempted for the density correlation functions, but with less success, in the absence of further approximation. To be concrete, let us consider the consequences of $p_t[\Phi]$ being *Gaussian*, for which the approximate equality in (4.57) becomes *exact*. That this is not a frivolous exercise in solving what we can solve but a representation of reasonable dynamics will be shown, in part, later, for the 'slow-roll' dynamics that we shall adopt. For the moment we take it for granted.

Specifically, suppose that (still for the one-dimensional case) Φ is a Gaussian field for which

$$\langle\Phi(x)\rangle_t = 0 = \langle\Phi(x)\Phi'(x)\rangle_t, \tag{4.61}$$

and that

$$\langle\Phi(x)\Phi(y)\rangle_t = W(|x-y|;t). \tag{4.62}$$

All other connected correlation functions are taken to be zero. Then all ensemble averages are given in terms of $W(r;t)$ which we have have seen to be the equal-time Wightman function,

$$W(|x-y|;t) = \langle\phi(x,t)\phi(y,t)\rangle \tag{4.63}$$

with the given initial conditions. In our case, where we shall assume thermal equilibrium initially, this is the usual thermal Wightman function. It is then straightforward to see that, if

$$f(r;t) = \frac{W(r;t)}{W(0;t)} \tag{4.64}$$

then

$$\bar{n}(t) = \frac{1}{\pi}(-f''(0;t))^{\frac{1}{2}}. \tag{4.65}$$

in agreement with (4.59). On using the same exponentiation that $\langle\delta[\Phi(x)]\rangle_t$ equals $\int d\!\!\!/\alpha\langle e^{i\alpha\Phi(x)}\rangle_t$ it takes only a little manipulation to cast the correlation function $C(|x|;t)$ of (2.8) in the form

$$C(r;t) = \frac{\partial h(r;t)}{\partial r}, \tag{4.66}$$

where

$$h(r;t) = \frac{-f'(r;t)}{2\pi\sqrt{1-f^2(r;t)}}. \tag{4.67}$$

If we are given $C(r;t)$, and wish to infer $f(r;t)$, and hence its power spectrum, we integrate equation (4.67) as

$$f(r;t) = \sin\left[\frac{\pi}{2} - 2\pi\int_0^r dr'\,h(r';t)\right] \tag{4.68}$$

in which $h(r;t)$ is, in turn, derived from $C(r;t)$ through equation (4.66). For example, for the Poisson distribution of zeroes given earlier in (2.19), we find

$$f(r;t) = sin\left(\frac{\pi}{2}\,e^{-2r/\xi(t)}\right) \tag{4.69}$$

However, the situation is usually the converse, with the dynamical model predicting $f(r;t)$, from which $C(r;t)$ follows. As a concrete example, suppose that

$$W(r;t) = \int d\!\!\!/k\, e^{ikx}\,\tilde{W}(k;t) \tag{4.70}$$

where

$$\tilde{W}(k;t) = \delta(k^2 - k_0^2(t)). \tag{4.71}$$

That is, there is only one wavelength in the model. Then

$$\begin{aligned} f(r;t) &= \cos\left(k_0(t)r\right), \\ \bar{n}(t) &= k_0(t)/\pi. \end{aligned} \tag{4.72}$$

The resulting

$$h(r;t) = \frac{k_0(t)}{2\pi}\mathrm{sign}(\sin(k_0(t)r)), \tag{4.73}$$

is exactly that of (2.18), in which the correlation function $C(r;t)$ is a sum of delta-functions, corresponding to a *regular* array of zeroes.

Now suppose that, instead of a single wavelength, $W(r;t)$ has contributions from frequencies peaked about k_0, as

$$\tilde{W}(k;t) = \frac{1}{2}\Big(\exp\{-\frac{1}{2}(k-k_0(t))^2/(\Delta k_0(t))^2\} + \exp\{-\frac{1}{2}(k+k_0(t))^2/(\Delta k_0(t))^2\}\Big) \tag{4.74}$$

In the limit $\Delta k_0(t) \to 0$ we recover the regular array but, for $\Delta k_0(t) \neq 0$,

$$W(r;t) = \int dk\, \cos kx\, \exp\{-\frac{1}{2}(k-k_0(t))^2/(\Delta k_0(t))^2\} \tag{4.75}$$

giving

$$f(r;t) = (\cos(k_0(t)r))\exp\{-\frac{1}{2}r^2(\Delta k_0(t))^2\}. \tag{4.76}$$

The corresponding $h(r;t)$ is no longer that of (2.18), but

$$h(r) = \frac{k_0\sin(k_0 r) + r(\Delta k_0)^2\cos(k_0 r)}{\sqrt{1-(\cos^2(k_0 r))\exp\{-r^2(\Delta k_0)^2\}}}\exp\{-\frac{1}{2}r^2(\Delta k_0)^2\}. \tag{4.77}$$

and we have dropped the time-labelling. A cursory inspection shows that $h(r)$ is damped periodic, and the broadening of its derivative $C(r;t)$ from its δ-function behaviour when $\Delta k_0(t) \neq 0$ measures the variance $\Delta\xi(t)$ in the separation $\xi(t)$ between adjacent zeroes. We see that, for small $\Delta k_0(t)/k_0(t)$,

$$\frac{\Delta\xi(t)}{\xi(t)} \propto \frac{\Delta k_0(t)}{k_0(t)}. \tag{4.78}$$

with a coefficient of proportionality of $O(1)$[8]. We note that, even when the variance $\Delta\xi(t)/\xi(t)$ is large, so that the zeroes appear much more random, the Gaussian behaviour is very different from the linear exponential behaviour of the Poisson distribution noted earlier. There is always correlation.

4.2 Monopoles

The extension to $O(2)$ line zeroes is messy, but leads to no surprises. Specifically, suppose that

$$\langle\Phi_a(\mathbf{x})\rangle_t = 0 = \langle\Phi_a(\mathbf{x})\partial_j\Phi_b(\mathbf{x})\rangle_t, \tag{4.79}$$

[8]It follows that the effect of the variance is to increase the density of zeroes to $\bar{n}(t) = \sqrt{k_0^2(t) + (\Delta k_0(t))^2}/\pi$.

and, further, that

$$\langle\Phi_a(\mathbf{x})\Phi_b(\mathbf{x}')\rangle_t = W_{ab}(|\mathbf{x}-\mathbf{x}'|;t) = \delta_{ab}W(|\mathbf{x}-\mathbf{x}'|;t), \tag{4.80}$$

is diagonal. As before, all other connected correlation functions are taken to be zero.

The density calculation proceeds as before. It follows[10, 11] that

$$\bar{n}(t) = \frac{1}{2\pi}(-f''(0;t)). \tag{4.81}$$

where derivatives are taken with respect to $r = |\mathbf{x}-\mathbf{x}'|$. The zero density- density correlation function is more complicated than for the kinks, albeit still in terms of $h(r;t)$ of (4.67), as[11]

$$C(r;t) = \frac{2}{r}h(r;t)h'(r;t). \tag{4.82}$$

If $C(r;t)$ is given, we can integrate equation (4.82) to obtain $f(r;t)$ as

$$f(r;t) = \sin\left[\frac{\pi}{2} - 2\pi\int_0^r dr\left(-\int_r^\infty dr'\,r'\,C(r';t)\right)^{1/2}\right], \tag{4.83}$$

on incorporating the conservation of topological charge.

However, we are more interested in calculating $C(r;t)$ for specific $f(r;t)$. As for kinks above, the simplest case corresponds to taking

$$W(r;t) = \int d\!\!\!/^2k\,e^{i\mathbf{k}.\mathbf{x}}\,\tilde{W}(\mathbf{k};t) \tag{4.84}$$

in which there is only one wavelength

$$\tilde{W}(\mathbf{k};t) = \delta(\mathbf{k}^2 - k_0^2(t)). \tag{4.85}$$

where $k_0(t)$ depends on time t. As a result

$$f(r;t) = J_0(k_0(t)r), \tag{4.86}$$

$$\bar{n}(t) = \frac{k_0^2(t)}{\pi} \tag{4.87}$$

and, from (4.67),

$$h(r;t) = \frac{J_1(k_0(t)r)}{2\pi\sqrt{1-J_0^2(k_0(t)r)}}. \tag{4.88}$$

Because of its rotational invariance $C(r;t)$ of (2.25) cannot show the δ-function behaviour of its counterpart on the line, but regularity is implicit in the strong regular oscillatory peaking of C that comes from the Bessel functions in the numerator.

If, instead of a single wavelength, $W(r;t)$ has contributions from frequencies peaked about k_0 with variance Δk_0, then the secondary peaking is reduced. Specifically, consider

$$W(r;t) = \int dk\,J_0(kr)\,\exp\{-\frac{1}{2}(k-k_0(t))^2/(\Delta k_0(t))^2\}. \tag{4.89}$$

In fact, unlike the one-dimensional case, for which $W(r;t)$ and $f(r;t)$ of (4.76) retain their oscillatory behaviour, when $\Delta k_0(t)/k_0(t)$ is large enough $W(r;t)$ does *not* oscillate or even change sign. This is understood as indicating a less regular distribution of monopoles. The dispersion in wavelength $\Delta k_0(t)$ leads to a variance $\Delta\xi$ in their separation that we expect to satisfy

$$\frac{\Delta\xi(t)}{\xi(t)} \propto \frac{\Delta k_0(t)}{k_0(t)}. \tag{4.90}$$

In itself this short-range behaviour of $f(r;t)$ is enough to give the variance in winding number through a surface of perimeter L as $(\Delta N)^2 = O(L)$ for the reasons given earlier.

4.3 Vortices

Finally, for line zeroes the Gaussian approximation has yet more complicated consequences. As for monopoles, we assume diagonal correlation functions

$$\langle\Phi_a(\mathbf{x})\Phi_b(\mathbf{x}')\rangle_t = W_{ab}(|\mathbf{x}-\mathbf{x}'|;t) = \delta_{ab}W(|\mathbf{x}-\mathbf{x}'|;t), \tag{4.91}$$

from which $f(r;t)$ can be defined as in (4.64), and vanishing field expectation value and and the independence of the field and its derivatives

$$\langle\Phi_a(\mathbf{x})\rangle_t = 0 = \langle\Phi_a(\mathbf{x})\partial_j\Phi_b(\mathbf{x})\rangle_t, \tag{4.92}$$

As might have been anticipated, $\bar{n}(t)$ is as for monopoles

$$\bar{n}(t) = \frac{1}{2\pi}(-f''(0;t)), \tag{4.93}$$

whereas[10, 11] the transverse and longitudinal parts of the density correlation (2.38) are, still in terms of $h(r;t)$,

$$A(r;t) = \frac{2}{r}h(r;t)h'(r;t) \tag{4.94}$$

(the same in form as $C(r;t)$ of (2.25)), and

$$B(r;t) = \frac{2}{r^2}h^2(r;t) > 0. \tag{4.95}$$

We note that B is *positive.* The conservation law (2.40) follows, as

$$\int d^3x\,(2A+B) \propto \int_0^\infty dr\,\frac{d}{dr}\Big(rh^2\Big) = 0 \tag{4.96}$$

Given either $A(r;t)$ or $B(r;t)$ we can reconstruct $f(r;t)$. For given $A(r;t)$, $f(r;t)$ has a similar form to that of (4.83) (except for the absence of charge conservation). More simply, for given $B(r;t)$,

$$f(r;t) = \sin\left[\frac{\pi}{2} - \pi\sqrt{2}\int_0^r dr'\,r'(2B(r';t))^{1/2}\right], \tag{4.97}$$

If, as before, we asssume a single wavelength, then

$$\begin{aligned} W(r;t) &= \int đ^3k\, e^{i\mathbf{k}.\mathbf{x}}\,\delta(\mathbf{k}^2 - k_0^2(t)) \\ &= \frac{\sin(k_0(t)r)}{k_0(t)r} \equiv \mathrm{sinc}\,(k_0(t)r) \end{aligned} \tag{4.98}$$

It follows that

$$\bar{n}(t) = \frac{k_0^2(t)}{6\pi} \tag{4.99}$$

and

$$h(r;t) = \frac{\sin(k_0(t)r) - k_0(t)r\,\cos(k_0(t)r)}{2\pi r\sqrt{k_0^2(t)r^2 - \sin^2(k_0(t)r)}} \tag{4.100}$$

from which $A(r;t)$ and $B(r;t)$ can be constructed from above. A and B again show periodic peaking, pointing to as regular a distribution of line-zeroes as possible. Specifically, when $k_0(t)r \gg 1$

$$A(r;t) \simeq \frac{-k_0(t)\sin 2k_0(t)r}{4\pi r^3} \tag{4.101}$$

and

$$B(r;t) \simeq \frac{\cos^2 k_0(t)r}{2\pi r^4} \tag{4.102}$$

up to non-leading terms. On the other hand, when $k_0(t)r \ll 1$ $A(r;t) = O(\bar{n}^2(t))$ and negative, whereas $B(r;t) = O(\bar{n}^2(t)/k_0^2(t)r^2)$.

Yet again, if, instead of a single wavelength, $W(r;t)$ has contributions from frequencies peaked about k_0 with variance $\Delta k_0(t)$ the secondary peaking is reduced. If we take

$$W(r;t) = \int dk\,\mathrm{sinc}(kr)\,\exp\{-\frac{1}{2}(k - k_0(t))^2/(\Delta k_0(t))^2\}. \tag{4.103}$$

then, as for monopole zeroes in two dimensions, once $\Delta k_0(t)/k_0(t)$ is sufficiently large, $W(r;t)$ ceases to oscillate or even to vanish. Yet again we understand the variance in k as inducing a variance $\Delta\xi$ in their separation approximately satisfying (4.90). However, the short-distance behaviour of A and B is unchanged qualitatively.

We shall see later, in our model making, how distributions approximately of the form (4.103) arise naturally. However, there is another, even simpler, and not wholly dissimilar way of introducing a dispersion about the wavenumber $k_0(t)$ that has been adopted in a cosmological context[6]. Because of the similarities between it and the model that we shall introduce later we shall look at it in some detail. The work of [6] essentially consists in taking $W(r;t)$ as

$$W^{(n)}(r;t) \propto \int_0^{k_0(t)} dk\,\mathrm{sinc}(kr)\left(\frac{k}{k_0(t)}\right)^{2+n} \tag{4.104}$$

describing fields with a variable power spectrum k^n, cut off at $k = k_0(t)$.

For $n \geq 0$ the behaviour of the normalised field correlation functions $f^{(n)}(r;t) = W^{(n)}(r;t)/W^{(n)}(0;t)$ is determined by the sharp cutoff, unlikely to be present in any realistic models. However, it follows that the limit $n \to \infty$ reproduces the single-mode result of (4.98). The density $\bar{n}(t)$ is calculated easily as

$$\bar{n}(t) = \left(\frac{3+n}{5+n}\right)\frac{k_0^2(t)}{6\pi} \tag{4.105}$$

reducing to (4.99) as $n \to \infty$, as it must.

As n decreases the power in long wavelength modes increases. For example, for $n = 0, -1, -2$

$$\begin{aligned} f^{(0)}(r;t) &= \frac{3}{(k_0(t)r)^3}(\sin(k_0(t)r) - k_0(t)r\,\cos(k_0(t)r) \\ f^{(-1)}(r;t) &= \frac{2}{(k_0(t)r)^2}(1-\cos(k_0(t)r)) \\ f^{(-2)}(r;t) &= \frac{1}{k_0(t)r}\,Si(k_0(t)r) \end{aligned} \tag{4.106}$$

The case $n = 0$ corresponds to white noise on scales larger than k_0^{-1}[9]. We see immediately that $f^{(-1)}(r;t)$ of (4.106) has double the period of the large-n $W(r;t)$ of (4.98), indicating a reduced density directly.

If we calculate the density-density anticorrelation function $A(r;t)$ for these $f^{(n)}$ and others we find that, for small r (where the effects of the sharp cutoff are less apparent) it can be written as

$$A(r;t) \approx -\bar{n}^2(t)(a_n - b_n(k_0(t)r)^2) \tag{4.107}$$

where a_n and b_n are positive and, by extracting a factor $\bar{n}^2(t)$ we are working at constant density. The strength a_n of the anticorrelation is approximately constant and $O(1)$ for $n > 0$, whereas for $n < 0$ it diminishes rapidly, vanishing when $n \approx -2$. For $n > -2$ the range of the anticorrelation is determined by b_n^{-1}. Over the range $-2 < n < \infty$, b_n is approximately linear, growing from approximately zero at $n = -2$ with slope $O(1)$. Thus, as n becomes increasingly negative (but $n > -2$) the range over which the strings have influence on one another increases, but with diminishing strength. For $n = -2$, $k_0(t)$ ceases to be a dominant wavenumber and there is no quantity to identify with a domain size. We would interpret this as implying the greatest variance in string-zero separations.

For large r, we note that

$$f^{(n)}(r;t) = O\left(\frac{1}{(k_0(t)r)^2}\right) \tag{4.108}$$

for integer $n \geq -1$, and $f^{(-2)}(r;t) = O(1/(k_0(t)r))$. The vanishing of $\bar{n}(t)$ at $n = -3$ is a consequence of the infrared divergence at this value, but can be ignored since it is difficult to see how values of $n < -2$ could arise from realistic dynamics.

[9] White noise on all scales gives a δ-function for W.

With these examples behind us we are now ready to attempt to make predictions in a simple dynamical model of Gaussian fluctuations.

5 Gaussian Dynamics from an Instantaneous Quench and its Coarsening

In practice, our ability to construct $p_t[\Phi]$ or, equivalently, the field correlation functions over the whole timescale $t > t_0$ from initial quantum fluctuations to late time classicality is severely limited. In particular, a Gaussian approximation can have only a limited applicability. To see what this is, it is convenient to divide time into four intervals, to each of which we adopt a different approach. With M setting the mass-scale, there is an initial period $t_0 < t < t_i = O(M^{-1})$, before which the field is able to respond to the quench, however rapidly it is implemented, and which we can largely ignore. Assuming weak coupling, of which more later, this is followed by an interval $t_i < t < t_{sp}$ in which, provided the quench is sufficiently rapid, domains in field phase form and grow. For the reasons given above vortices (or monopoles) will appear and be driven apart by the coalescence of these domains. For the quartic winebottle potential $V(\phi)$ with minima at $|\phi| = \phi_0$ domain growth begins to stop once the field has reached its spinodal value $|\phi|^2 = \phi_0^2/3 = O(M^2/\lambda)$ at the ring of inflection $V''(\phi) = 0$. This occurs at times peaked around some $t = t_{sp}$. To estimate t_{sp} we observe that, for short times and weak coupling, the long wavelength modes with wavenumber $k \approx 0$ grow as $\tilde{\phi}_{\mathbf{k}}(t) = O(Me^{Mt})$ as they fall from the top of the potential hill. Thus t_{sp} is given in terms of the coupling strength by

$$\exp\{-2Mt_{sp}\} = O(\lambda) \tag{5.109}$$

or equivalently, $t_{sp} = O(M^{-1}\ln(1/\lambda))$. We assume that λ is sufficiently small that t_{sp} is significantly larger than t_i. The defects are beginning to freeze in and fluctuations are now too small to undo them. In the third period, beginning at t_{sp}, the field magnitude relaxes dissipatively to the ground state values and the vortices complete their freezing in. Finally, in the last period, the vortices behave semiclassically.

We shall see that our ability to calculate from first principles is limited to the second period $t_i < t < t_{sp}$. On the (as yet unproven) assumption that the distribution of relevant zeroes at time t_{sp} is left largely unchanged by their final freezing in, this distribution of vortices can then be taken as initial data for the final evolution of the network. Similar considerations apply to monopoles.

It is not difficult to justify our adoption of Gaussian field fluctuations for this second period $t_i < t < t_{sp}$ of vortex production. We have already assumed that the initial conditions correspond to a disordered state. In the absence of any compelling evidence to the contrary we achieve this by adopting thermal equilibrium at a temperature T higher than the critical temperature T_c for $t < t_0$, as we anticipated earlier. That is, the

action S_0 of (3.50) that characterises the initial conditions is

$$S_0[\phi_3] = \int d^{D+1}x \Big[\frac{1}{2}(\partial_\mu \phi_{3\,a})(\partial^\mu \phi_{3\,a}) - \frac{1}{2}m(T)^2\phi_{3\,a}^2 - \frac{1}{4}\lambda_0(\phi_{3\,a}^2)^2\Big]. \tag{5.110}$$

with $m^2(T) > 0$. If λ_0 is weak the resulting field distribution is approximately Gaussian, and it is little loss to take it to be exactly Gaussian, $\lambda_0 = 0$. As we shall see later, initial conditions generally give slowly varying behaviour in the correlation function, in contrast to the rapid variation due to the subsequent dynamics, and calculations are insensitive to them.

In order to have as simple a change of phase as possible, we assume an idealised quench, in which, at $t = t_0$, $m^2(t)$ changes sign everywhere. Most simply, this change in sign in $m^2(t)$ can be interpreted as due to a reduction in temperature. Even more, at first we further simplify our calculation by assuming that, for $t > t_0$, $m^2(t)$ takes the *negative* value $m^2(t) = -M^2 < 0$ *immediately*, where $-M^2$ is the mass parameter of the (cold) relativistic Lagrangian. That is, the potential at the origin has been *instantaneously* inverted, breaking the global $O(2)$ symmetry. If, as we shall further assume, the $\lambda|\phi|^4$ field coupling is very weak then, for times $M(t - t_0) < \ln(1/\lambda)$, the ϕ-field, falling down the hill away from the metastable vacuum, will not yet have experienced the upturn of the potential, before the point of inflection and we can set $\lambda = 0$. Thus, for these small time intervals, $p_t[\Phi]$ is Gaussian, as required. Henceforth, we take $t_0 = 0$.

For such a weakly coupled field the onset of the phase transition at time $t = 0$ is characterised by the instabilities of long wavelength fluctuations permitting the growth of correlations. Although the initial value of $\langle\phi\rangle$ over any volume is zero, we anticipate that the resulting evolution will lead to domains of constant ϕ phase, whose boundaries will trap vortices. This situation of inverted harmonic oscillators was studied many years ago by Guth and Pi[13] and Weinberg and Wu[14]. In the context of domain formation, we refer to the recent work of Boyanovsky et al.[16], and our own[4].

Consider small amplitude fluctuations of ϕ_a, at the top of the parabolic potential hill. Long wavelength fluctuations, for which $|\mathbf{k}|^2 < M^2$, begin to grow exponentially. If their growth rate $\Omega(k) = \sqrt{M^2 - |\mathbf{k}|^2}$ is much slower than the rate of change of the environment, then those long wavelength modes are unable to track the quench. Unsurprisingly, the time-scale at which domains appear in this instantaneous quench is $t_i = O(M^{-1})$. As long as the time taken to implement the quench is comparable to t_i and less than $t_{sp} = O(M^{-1}\ln(1/\lambda))$ the approximation is relevant. We shall relax the condition in the next section.

For the moment we ignore the effect of the interactions that stabilise the potential and permit the zeroes corresponding to the defects to freeze in. In this free-roll period, $W(r;t)$ has to be built from the modes $\mathcal{U}^{\pm}_{a,k}$, $\mathcal{U}^{+}_{a,k} = (\mathcal{U}^{-}_{a,k})^*$, satisfying the equations of motion

$$\Big[\frac{d^2}{dt^2} + \mathbf{k}^2 + m^2(t)\Big]\mathcal{U}^{\pm}_{a,k} = 0, \tag{5.111}$$

For the idealised case proposed above $m^2(t)$ is of the form

$$\begin{aligned} m^2(t) &= m_0^2 > 0 \text{ if } t < 0, \\ &= -M^2 < 0 \text{ if } t > 0. \end{aligned} \tag{5.112}$$

If we make a separation into the unstable long wavelength modes, for which $|\mathbf{k}| < M$, and the short wavelength modes $|\mathbf{k}| > M$, then $W(r;t)$ is the real quantity

$$W(r;t) = \int d\!\!\!/^D k \, e^{i\mathbf{k}.\mathbf{x}} C(k) \mathcal{U}^+_{a,k}(t) \mathcal{U}^-_{a,k}(t) \tag{5.113}$$

$$\begin{aligned} &= \int_{|\mathbf{k}|<M} d\!\!\!/^D k \, e^{i\mathbf{k}.\mathbf{x}} C(k) \Big[1 + A(k)(\cosh(2\Omega(k)t) - 1) \Big] \\ &+ \int_{|\mathbf{k}|>M} d\!\!\!/^D k \, e^{i\mathbf{k}.\mathbf{x}} C(k) \Big[1 + a(k)(\cos(2w(k)t) - 1) \Big] \end{aligned} \tag{5.114}$$

with no summation over a, $r = |\mathbf{x}|$ and

$$\begin{aligned} \Omega^2(k) &= M^2 - |\mathbf{k}|^2, \\ w^2(k) &= -M^2 + |\mathbf{k}|^2, \\ A(k) &= \frac{1}{2}\Big(1 + \frac{\omega^2(k)}{\Omega^2(k)}\Big), \\ a(k) &= \frac{1}{2}\Big(1 - \frac{\omega^2(k)}{w^2(k)}\Big). \end{aligned} \tag{5.115}$$

The initial conditions are encoded in $\mathcal{C}(k)$, which takes the familiar form

$$\mathcal{C}(k) = \frac{1}{2\omega(k)} \coth(\beta_0 \omega(k)/2) \tag{5.116}$$

in which $\omega^2(k) = |\mathbf{k}|^2 + m_0^2$.

In the single-mode and other approximations adopted earlier the folly of counting *all* zeroes was not apparent. It is now, in the presence of the ultraviolet divergence of $W(r;t)$ at $r = 0$ in all dimensions. None of the expressions given above is well-defined. To identify which zeroes will turn into our vortex network requires coarse-graining.

The way to do this is determined by the dynamics. Firstly, we note that if Φ is Gaussian, then so is the coarsegrained field on scale L,

$$\Phi_L(\mathbf{x}) = \int d^D x' \, I(|\mathbf{x} - \mathbf{x}'|) \Phi(\mathbf{x}'), \tag{5.117}$$

where $I(r)$ is an indicator (window) function, normalised to unity, which falls off rapidly for $r > L$. The only change is that (4.80) is now replaced by

$$\langle \Phi_{L,a}(\mathbf{x}) \Phi_{L,b}(\mathbf{x}') \rangle_t = W_{L,ab}(|\mathbf{x} - \mathbf{x}'|; t) = \delta_{ab} W_L(|\mathbf{x} - \mathbf{x}'|; t), \tag{5.118}$$

where $W_L = \int\int IWI$ is now cut off at distance scale L. $W_L(0;t)$, its derivatives, and all relevant quantities constructed from W_L are ultraviolet *finite*. The distribution of zeroes, or line zeroes, of Φ_L is given in terms of W_L as in the previous section.

Choosing $L = M^{-1}$ solves all our problems simultaneously. At wavelengths $k^{-1} < L$ ($i.e. k > M$) the field fluctuations are oscillatory, with time scales $O(M^{-1})$. Only those long wavelength fluctuations with $k^{-1} > L$ have the steady exponential growth that can lead to the field migrating on larger scales to its groundstates. Further, as the field settles to its groundstates, the typical vortex thickness is $O(M^{-1})$, and we only wish to attribute *one* zero to each vortex cross-section (or one zero to each monopole). By taking $L = M^{-1}$ we are choosing not to count zeroes within a string, apart from the central core. We shall see later that it is not crucial to take $L = M^{-1}$ exactly, but sufficient to take $L = O(M^{-1})$.

We are now in a position to evaluate $p_t[\Phi]$, or rather $W_L(r;t)$, which we now write as $W_M(r;t)$ for $t > 0$, where

$$W_M(r;t) = \int_{|\mathbf{k}|<M} d\!\!\!{}^-{}^D k\, e^{i\mathbf{k}.\mathbf{x}} \mathcal{C}(k)\Big[1 + A(k)(\cosh(2\Omega(k)t) - 1)\Big] \tag{5.119}$$

or,equivalently

$$W_M(r;t) = \int_{|\mathbf{k}|<M} d\!\!\!{}^-{}^D k\, e^{i\mathbf{k}.\mathbf{x}} \mathcal{C}(k)\Big[1 + \frac{1}{2}A(k)(e^{\Omega(k)t} - e^{-\Omega(k)t})^2\Big] \tag{5.120}$$

and calculate the density of zeroes accordingly. Even though the approximation is only valid for small times, there is a regime $Mt \geq 1$, for small couplings, in which t is large enough for $e^{Mt} \gg 1$ and yet Mt is still smaller than $Mt_{sp} = O(\ln(1/\lambda))$ when the fluctuations begin to reach the spinodal point on their way to the ground-state manifold. In this regime the exponential term in the integrand dominates and

$$\begin{aligned} W_M(r;t) &\simeq \frac{1}{2}\int_{|\mathbf{k}|<M} d\!\!\!{}^-{}^D k\, \mathcal{C}(k)A(k)e^{i\mathbf{k}.\mathbf{x}}\, e^{2\Omega(k)t} && (5.121)\\ &= \frac{1}{2(2\pi)^D}\int_{|\mathbf{k}|<M} d\Omega\, dk\, k^{D-1}\, e^{2\Omega(k)t}\,\mathcal{C}(k)A(k)e^{i\mathbf{k}.\mathbf{x}} && (5.122)\end{aligned}$$

For the moment we assume that M and m_0 are comparable. In our context of initial thermal equilibrium at temperature T_0 this corresponds to beginning the quench from well above the transition if, as earlier, we identify m_0 with the thermal mass at temperature T_0, most simply approximated to one loop by

$$m_0^2 = -M^2\Big(1 - \frac{T_0^2}{T_c^2}\Big) \tag{5.123}$$

where $T_c = O(M/\sqrt{\lambda})$ in the same approximation. The definition (5.123) needs some qualifications, but for the moment we accept it as it stands . We shall consider the effect of modifying it later. For $T_0 > T_c$, and weak coupling $\mathcal{C}(k)$ of (5.116) is approximately

$$\mathcal{C}(k) = \frac{T_0}{\mathbf{k}^2 + m_0^2}. \tag{5.124}$$

The integral at time t is then dominated by the peak in $k^{D-1}e^{2\Omega(k)t}$ at k around k_0, where

$$tk_0^2 = \frac{D-1}{2}M\Big(1+O\Big(\frac{1}{Mt}\Big)\Big). \tag{5.125}$$

To check the consistency of the assumption that $\mathcal{C}(k)$ and $A(k)$ are slowly varying in comparison to this peak we observe that, at the largest times of interest to us, $t = O(t_{sp})$

$$k_0^2 = O\Big(\frac{M}{t_{sp}}\Big) \ll M^2 \tag{5.126}$$

and thereby, for a quench from well above the transition, equally less than m_0^2. $\mathcal{C}(k)$ of (5.124) can be approximated by

$$\mathcal{C}(k) = \frac{T_0}{m_0^2}. \tag{5.127}$$

The effect of changing the initial thermal conditions is only visible in the $O(1/Mt)$ term. Since the overall normalisation of $W(r;t)$ is irrelevant, both $\mathcal{C}(k)$ and $A(k) \approx 1$ can be factored out. For weak coupling we recover what would have been our first naive guess for the coarse-grained correlation function $\langle\phi_{L,a}(\mathbf{r},t)\phi_{L,b}(\mathbf{r}',t)\rangle$ based on the growth of the unstable modes $\phi_a(\mathbf{k},t) \simeq e^{\Omega(k)t}$ alone,

$$W_M(r;t) \propto \int_{|\mathbf{k}|<M} đ^D k\, e^{i\mathbf{k}.\mathbf{x}}\, e^{2\Omega(k)t}, \tag{5.128}$$

provided we have not quenched from too close to the transition and $t > M^{-1}$ (preferably $t \gg M^{-1}$). It is in this sense that our conclusions are insensitive to the initial conditions. However, we might hope that this insensitivity extends to non-thermal initial conditions as long as we are far from the transition.

We note that if we had coarse-grained to *longer* wavelengths $|\mathbf{k}| < M_0 < M$ then, once $Mt > M^2/M_0^2$ the dominant peak lies inside the integration region and the results are insensitive to the value of the cutoff. Provided $M/M_0 \simeq 1$ this will be the case because of (5.126).

On approximating the peak around k_0 by a Gaussian (which assumes this insensitivity to coarse-graining on somewhat larger scales) we see that, for monopoles in two dimensions $W_M(r;t)$ has the form of (4.89),

$$W_M(r;t) \propto \int dk\, J_0(kr)\, \exp\{-\frac{1}{2}(k-k_0(t))^2/(\Delta k_0(t))^2\}, \tag{5.129}$$

where $k_0(t)$ is given by (5.125), and

$$\frac{\Delta k_0(t)}{k_0(t)} = \frac{1}{\sqrt{2}}. \tag{5.130}$$

Further, for the more interesting case of vortices in $D = 3$ dimensions, $W_M(r;t)$ can be approximated by (4.103),

$$W_M(r;t) \propto \int dk\, \mathrm{sinc}(kr)\, \exp\{-\frac{1}{2}(k-k_0(t))^2/(\Delta k_0(t))^2\}, \tag{5.131}$$

where

$$\frac{\Delta k_0(t)}{k_0(t)} = \frac{1}{2} \tag{5.132}$$

is also large. We interpret (5.130) and (5.132) as implying large variance in the separation of the zeroes that are defining our defects.

Although the forms (5.129) and (5.131) represent the spread in wavenumber well they do so at the expense of a correct description of the long wavelength modes. $W_M(r;t)$ of (5.128) can be compared directly to the empirical form of (4.104) by writing it (up to the usual arbitrary normalisation) as

$$W_M(r;t) \simeq \int_{|\mathbf{k}|<M} dk\, \mathrm{sinc}(kr)\Big(\frac{k}{k_0(t)}\Big)^2 e^{2\Omega(k)t}. \tag{5.133}$$

In the terminology of (4.104) this shows that, for long wavelengths, the power is $n = 0$, determined entirely by the radial k^{D-1} behaviour in $D = 3$ dimensions. Insofar as distributions of vortices are determined by the power in long wavelengths the peak at $k = k_0(t)$ may, to some extent, be approximated by the cutoff at $k_0(t)$ of (4.104). We shall return to this later.

Rather than evaluate correlation functions from (5.128) directly it is convenient to approximate it differently. It is not difficult to see that, for the values of $\Delta k_0(t)/k_0(t)$ given above, $W_M(r;t)$ approximately falls monotonically in r to zero from above (any oscillatory behaviour is of very small amplitude). It is a good approximation if, in (5.128) we expand $\Omega(k)t$ as

$$\Omega(k)t \approx Mt - \frac{k^2 t}{2M} \tag{5.134}$$

from which

$$W_M(r;t) \simeq e^{2Mt} \int_{|\mathbf{k}|<M} d\!\!{}^-{}^D k\, e^{i\mathbf{k}.\mathbf{x}}\, e^{-k^2 t/M}. \tag{5.135}$$

This approximation maintains the peak in the integrand at $k_0(t)$ of (5.125). Taking it as it stands gives

$$f(r;t) = \exp\{-r^2 M/4t\}, \tag{5.136}$$

independent of dimension, on dropping the upper integration bound[10]. As such it correctly reproduces the small-r behaviour of $f(r;t)$ (and $f^0(r;t)$ of (4.106)) and, although the simple Gaussian behaviour breaks down for $r^2 M/4t \gg 1$, the rapid large-r falloff is qualitatively correct.

We conclude with one more observation. Even in the disordered state for $t < 0$ there were zeroes (or lines of zeroes) induced by thermal fluctuations. Effectively, the equal-time two-point correlation function for $t < 0$ is

$$W_M(r;t) \simeq T_0 \int_{|\mathbf{k}|<M} d\!\!{}^-{}^D k\, \frac{e^{i\mathbf{k}.\mathbf{x}}}{\mathbf{k}^2 + m_0^2} = \int_{|\mathbf{k}|<M} d\!\!{}^-{}^D k\, e^{i\mathbf{k}.\mathbf{x}} \mathcal{C}(k). \tag{5.137}$$

[10] This is not a question of dropping coarse-graining but merely approximating the integral.

Because of equilibrium there is no time-dependence in $W_M(r;t)$ of (5.137). There is an implicit cutoff in (5.137) at the thermal wavelength $\beta_0 = T_0^{-1}$, but we are not interested in zeroes on a smaller scale than the same field correlation length M^{-1} that determines defect size. In fact, assuming $m_0 \approx M$, it makes little difference whether we coarsegrain to $|\mathbf{k}| < M$ or to $|\mathbf{k}| < m_0$.

Whatever, the surface density of zeroes is

$$\bar{n}(t) = O(m_0^2) = O(M^2). \tag{5.138}$$

However, we should not think of these zeroes as the precursors of the defects that appear after the transition. They are totally transient. We see this by the presence of oscillatory factors $e^{i\omega(k)\,\Delta t}$ in the two-point correlation function at unequal times $t, t+\Delta t$. In fact, the term (5.137) is the first term in (5.119) and (5.120), in fact the *only* term in either of these expressions when $t = 0$. The reason for rewriting (5.119) as (5.120) was to discriminate between this term (the 1 in the square bracket of (5.119)) and the term $-A(k) \approx -1$, in the same bracket, whose origin is the interference between exponentially increasing and decreasing terms in the mode evlution and has nothing to do with thermal fluctuations directly. Since the normalisation of $W_M(r;t)$ has no effect on the density and distributions of zeroes and line zeroes these thermal fluctuations are suppressed, relative to the long wavelength peak that shows the growth of domains, by a factor $O(e^{-2Mt})$. Thus, although the thermal fluctuations remain as strong in absolute terms, their contributions to the counting of zeroes, and hence the creation of defects, vanishes rapidly, which is why we did not include them earlier.

Similarly, had we chosen to coarse-grain on a somewhat shorter scale $|\mathbf{k}| < M_0$, where $M_0 > M$, then $W(r;t)$ would have acquired some (finite) oscillatory terms from (5.114) that in turn would have been suppressed, relative to the long wavelength peak, by a factor $O(e^{-2Mt})$. For large enough times t_{sp} they would play no contribution, provided $M/M_0 \simeq 1$. Taken with our earlier observations on coarse-graining at larger scales than M^{-1} we see that coarse-graining on a scale comparable to the thickness of cold defects is all that is required, with fine-tuning being unnecessary.

6 Slowing the Quench

Our adoption of an instantaneous quench, in which $m^2(t)$ changes as in (5.112) is obviously unrealistic. Any change in the environment requires some time τ to implement. In the context of the flat spacetime free-field approximation that we have adopted so far we should replace $m^2(t)$ of (5.112) by an effective $(mass)^2$ that interpolates between m_0^2 and its cold value $-M^2$ over τ. If τ were so small that $M\tau < 1$ we expect no significant change, since the field would not have responded in the time available anyway. However, once

$$1 < M\tau < Mt_{sp} \tag{6.139}$$

there will be an effect. As the wavenumber k of the field modes increases towards M the time available for their growth is progressively reduced. However, the long wavelength

modes have the same time to grow as before. As a result, the peaking of the integral of $W(r;t)$ will occur at a smaller value of k for the same time t after the quench is begun, leading to a lower density of defects than would have occurred otherwise. Alternatively, the field is correlated over larger distances than would have happened otherwise.

To see how this occurs quantitatively it is not necessary to go beyond simple approximations, in the absence of any compelling reason to make a specific choice for $m^2(t)$. [Such a reason occurs if we have a definite spacetime metric driving the transition, as happens in inflationary models[13, 18]].

Consider the behaviour

$$\begin{aligned} m^2(t) &= m^2, \quad t < 0 \\ &= -M^2\frac{t}{\tau}, \ 0 \leq t < \tau \\ &= -M^2, \ \tau \leq t \end{aligned} \tag{6.140}$$

in which the quench is begun at time $t = 0$ and completed at time $t = \tau$. The rate of the quench, τ^{-1} is assumed large, $\tau < t_{sp}$. That is,

$$1 \leq M\tau < \ln(1/\lambda), \tag{6.141}$$

so that the quench is complete before the field has experienced the turnup of the potential towards its minima. As $\tau \to 0$ we recover the instantaneous behaviour examined earlier.

If the behaviour presented in (6.140) seems a little artificial in how the behaviour before $t = 0$ and after $t = 0$ are connected we note that, from our previous discussion, the behaviour of $m^2(t)$ for $t \leq 0$ is largely irrelevant, as long as m^2 is comparable to M^2. Only the behaviour for $t > 0$ will be important.

The calculation of $f(r;t)$ now requires a proper solution of the mode equation (5.111) for $m^2(t)$ of (6.140), in terms of which $W(r;t)$ is again given by (5.113). This is not possible to derive analytically, but it is not difficult to make analytic approximations. As before we coarse-grain $W(r;t)$ to eliminate oscillatory modes, bounding $|\mathbf{k}|$ by M. Whereas modes of all wavelengths $|\mathbf{k}| < M$ begin to grow exponentially at the same time $t = 0$ for an instantaneous quench ($\tau = 0$), for $\tau \neq 0$ modes of wavenumber k do not begin to grow until time $t_\tau(k)$, at which

$$k^2 = -m^2(t_\tau(k)) \tag{6.142}$$

For the choice of m^2 above in (6.140)

$$t_\tau(k) = \frac{\tau k^2}{M^2} \tag{6.143}$$

As we noted earlier, although the very long wavelength growth is activated at $t \approx 0$, the shorter the wavelength (but still $|\mathbf{k}| < M$) the shorter the time the modes have available before their switchoff at $t = t_{sp} = O(M^{-1}\ln(1/\lambda))$.

A crude, but helpful approximation, for times $M\tau < Mt < ln(1/\lambda)$ is to mimic this effect by modifying (5.128) as

$$W_{M,\tau}(r;t) \simeq \int_{|\mathbf{k}|<M} d\!\!\!{}^{-}{}^D k\, e^{i\mathbf{k}.\mathbf{x}}\ e^{2\Omega(k)(t-t_\tau(k))}, \tag{6.144}$$

In the further approximation of (5.134), in which we expand in powers of k,

$$\begin{aligned} W_{M,\tau}(r;t) &\simeq \int_{|\mathbf{k}|<M} d\!\!\!{}^{-}{}^D k\, e^{i\mathbf{k}.\mathbf{x}}\ e^{(t-t_\tau(k))(2M-k^2/M)} \\ &= e^{2Mt}\int_{|\mathbf{k}|<M} d\!\!\!{}^{-}{}^D k\, e^{i\mathbf{k}.\mathbf{x}}\ e^{-k^2(t+2\tau)/M}\Big[1+O(\frac{\tau k^4}{M^3})\Big]. \end{aligned} \tag{6.145}$$

On comparing $W_{M,\tau}(r;t)$ of (6.145) to that of (5.135) it follows that, at the level of the Gaussian approximation, the effect of slowing the quench is to replace $f(r;t)$ of (7.149) by

$$f_\tau(r;t) \approx \exp\{-r^2 M/4t_\tau\} \tag{6.146}$$

where

$$t_\tau = t + 2\tau + O\Big(\frac{\tau}{M(t+2\tau)}\Big) \tag{6.147}$$

That is, in the first instance the effect of slowing the quench is no more than to reproduce the behaviour of an instantaneous quench, displaced in time by an interval 2τ. A more careful calculation would give slightly different answers, but the qualitative conclusion that slowing the quench reduces the defect density and hence makes the system look as if it had begun earler, is correct. In the absence of any reason to make a particular choice of quench, the result (6.147) is adequate for our purposes.

7 Freezing in the Vortices

We understand the dominance of wavevectors about $k_0^2 = M/t_\tau$ in the integrand of $W(r;t)$ as defining a length scale

$$\xi_\tau(t) = O(\sqrt{t_\tau/M}), \tag{7.148}$$

once $Mt > 1$, over which the independently varying fields ϕ_a are correlated in magnitude. To be specific, let us take $\xi_\tau(t) = 2\sqrt{t_\tau/M}$, whence

$$\begin{aligned} f_\tau(r;t) &= \exp\{-r^2 M/4t_\tau\} \\ &= \exp\{-r^2/\xi_\tau^2(t)\}. \end{aligned} \tag{7.149}$$

With this definition, the number density of line zeroes at early times is calculable from (4.81) as [11]

$$\bar{n}(t) = \frac{1}{\pi}\frac{1}{\xi_\tau(t)^2}, \tag{7.150}$$

[11] The density of monopoles in D=2 dimensions is identical.

both for monopoles and vortices, permitting us to interpret $\xi_{\tau}(t)$ as a correlation length or, equivalently, a domain size for phases. Although the zeroes and line zeroes have yet to freeze in as defects, this density of one potential defect per few correlation areas is commensurate with the Kibble mechanism cited earlier.

The zeroes that we have been tracking so far cannot yet be identified with the vortices (and monopoles) which provide the semiclassical network on the completion of the transition because the field has not achieved its groundstate values. In fact, and of greater importance at this stage, it is not even uniform in magnitude. Insofar as a classical picture is valid, the thermal fluctuations have kicked the field off the top of the upturned potential hill in different directions at slightly different times. The work of Guth and Pi[13] shows that the variance in this time is $\Delta t = O(M^{-1})$. Thus, even if the potential were a pure upturned parabola and the fields were to stop growing instantaneously at the moment that they reached the spinodal value $V''(\phi_{sp}) = 0$ (defined in terms of the non-Gaussian physical potential) it would take a further time $O(\Delta t)$ before the field caught up in all places.

An instantaneous halt to an increasing field growth is a huge oversimplification. More realistically, as the fields approach their spinodal values the effective $(mass)^2$ driving the expansion of the unstable modes decreases and the expansion of the domains slows. The energy of the fields in the long wavelength modes will cause an overshoot towards the potential bottom that, in the absence of dissipation, will be followed by a rebound, and perhaps a repeat (or many repeats) of the whole cycle[16]. The details will depend on the dissipation that the long wavelength modes endure. This dissipation is necessary to enable the fields to relax to their ground-state values, and for realistic fields has several sources. In particular, the act of coarse-graining induces dissipation[17], as would the presence of other fields[19] [12].

In this excursionary paper we shall just consider the rudiments of the effect of the field self-interaction on slowing down and stopping domain growth in an approximation in which dissipation is assumed to set in rapidly at the spinodal field values. At the simplest level this is, in some respects, like that of slowing the quench that we discussed previously, except that it occurs at the end of the period of interest, rather than the beginning. As there, the long wavelength modes are the least affected, the shorter wavelength modes (but still with $k < M$) having less time to grow. For this reason the density of zeroes, now identifiable as defects, is reduced.

As we have said, a full analysis is beyond the scope of this paper but, as with the slow quench, it is possible to provide a primitive approximation in which we can see this explicitly. This is enough for our present purposes, for which the power in the long wavelength modes will turn out to be the most relevant property that we need, and we hope that the crudity of our approximations leaves this untouched.

If we wish to retain the Gaussian approximation for the field correlation functions the

[12] A further source of dissipation is an expanding metric, inevitable in cosmological models. This is beyond us here.

best we can do is a mean-field approximation, or something similar[13]. In this approximation equations of motion are linearised by the substitution

$$(\phi_1^2 + \phi_2^2)\phi_1 \approx (3\langle\phi_1^2\rangle + \langle\phi_2^2\rangle)\phi_1 \tag{7.151}$$

and similarly for ϕ_2. Because of the diagonal nature of W_{ab} this means that

$$(\phi_1^2 + \phi_2^2)\phi_a \approx 4W(0;t)\phi_a \tag{7.152}$$

$W(r;t)$ still has the mode decomposition of (5.113), but the modes $\mathcal{U}_{a,k}^{\pm}$ now satisfy the equation (ignoring subtractions)

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 + m^2(t) + 4\lambda \int d\!\!\!{}^-{}^D p\, C(p)\mathcal{U}_{a,p}^{+}(t)\mathcal{U}_{a,p}^{-}(t)\right]\mathcal{U}_{a,k}^{\pm} = 0, \tag{7.153}$$

A detailed discussion of (7.153) will be given elsewhere, but a rough estimate of the effects of the interactions can be obtained by ignoring self-consistency and retaining only the unstable modes in the integral. For simplicity we revert to an instantaneous quench $m^2(t) = -M^2(t)$ for $t > 0$. On using the definition for λ in (5.109), we might approximate (7.153) in turn by

$$\left[\frac{d^2}{dt^2} + \mathbf{k}^2 - M^2 + M^2\, e^{2M(t-t_{sp})}\right]\mathcal{U}_{a,k}^{\pm} = 0, \tag{7.154}$$

where, further, we have ignored all but the dominant exponential behaviour due to a free-field roll. Note that the λ-coupling strength has been absorbed in the definition of t_{sp}, most justifiable for small λ. Equation (7.154) has the correct qualitative behaviour in that there are growing modes only for $t < t_{sp}$, after which all modes are oscillatory.

In this approximation a mode $\mathcal{U}_{a,k}^{\pm}$ can only grow until time $t_f(k)$, defined by

$$k^2 = M^2\left(1 - e^{2M(t_f(k)-t_{sp})}\right). \tag{7.155}$$

As anticipated, shorter wavelengths have less time to grow. Linearising (7.155) in the vicinity of t_{sp} as

$$k^2 = 2M^3(t_f(k) - t_{sp}) \tag{7.156}$$

shows that, beginning from an instantaneous quench, the mode with wavenumber k stops growing at time

$$\Delta t(k) \approx \frac{k^2}{2M^3} \tag{7.157}$$

before the modes of longest wavelength stop.

Instead of the instantaneous quench, now let us reintroduce a slower quench that takes time τ to implement, along the lines of (6.140) and the approximations that followed from

[13] But not a large-N $O(N)$ limit, since we only have simple defects in $D = N$ and $D = N + 1$ dimensions.

it. At the simplest level, the effect of the back-reaction on the field correlation function $W_{M,\tau}(r;t)$ of (6.144) is to replace it by

$$W_{M,\tau}(r;t) \simeq \int_{|\mathbf{k}|<M} đ^D k \, e^{i\mathbf{k}.\mathbf{x}} \; e^{2\Omega(k)(t-t^*(k))}, \tag{7.158}$$

where

$$\begin{aligned} t^*(k) &\approx t_\tau(k) + \Delta t(k) \\ &= \frac{\tau k^2}{M^2} + \frac{k^2}{2M^3}. \end{aligned} \tag{7.159}$$

In the further approximation of (5.134), in which we expand in powers of k,

$$W_{M,\tau}(r;t) \simeq \int_{|\mathbf{k}|<M} đ^D k \, e^{i\mathbf{k}.\mathbf{x}} \; e^{(t-t^*(k))(2M-k^2/M)}. \tag{7.160}$$

From this we derive an $f(r;t)$ of the form

$$f(r;t) \approx \exp\{-r^2 M/4t_\tau\} \tag{7.161}$$

where t_τ is now given by

$$t_\tau \approx t + 2\tau + \frac{1}{M}. \tag{7.162}$$

rather than by (6.147). We stress that, even at this level of approximation (in which any term can be multiplied by a coefficient $O(1)$), we have assumed sufficiently weak coupling and a rapid enough quench that

$$\tau + \frac{1}{2M} < t_{sp} \tag{7.163}$$

so that even the shortest wavelength considered has some time to grow. The details of (7.161) and (7.162) should not be taken too seriously, but the qualitative result that the effect of slowing the quench is to give a time-delay of $O(\tau)$, and the effect of back-reaction is to give a time-delay of $O(M^{-1})$ is as we would expect. A more realistic calculation[16] based on (7.153) would show that the field overshoots its spinodal values, retreats and overshoots again in damped oscillations. However, the small-r behaviour of (7.161) is good enough for our immediate purposes. We shall return to the full equation (7.153) later.

8 Densities and Correlations from a Quench

In the approximation given above the effect of slowing the quench (and taking the back-reaction into account) is just to increase the correlation length and thereby reduce the density of zeroes to one appropriate to an instantaneous transition that happened earlier and in which the field spontaneously stopped growing. The only way the zero-density can

decrease without the background space-time expanding is by zero-antizero annihilation. For monopoles this is simple removal of zeroes by superposition. For vortices it will include the collapse of small loops of zeroes. The end result is to preserve roughly one string zero per coherence area, a long held belief for whatever mechanism.

Let us perform all our calculations at $t = t_{sp}$, for which $\xi_\tau(t_{sp}) = \xi_{sp}$. For small enough λ, the density at $t = t_{sp}$ is, with our previous caveats about coefficients,

$$\bar{n}(t_{sp}) = \frac{1}{\pi}\frac{1}{\xi_{sp}^2} \approx \frac{1}{4\pi}\frac{M^2}{(Mt_{sp} + 2M\tau + 1)} \ll M^2, \tag{8.164}$$

We have retained the last term in the denominator as a reminder that the approximation may work even when $Mt_{sp} > 1$, but not too large. This shows that the line zeroes (or monopole zeroes) only create a small fraction of space as false vacuum. Equivalently, the typical separation of line zeroes (or monopole zeroes) is significantly larger than the thickness of a cold vortex (or monopole) once they have frozen in. This is one reason why we can begin to consider these line zeroes (and point zeroes) as serious candidates for vortices (and monopoles) even though the field has not fully relaxed to its ground states.

A further reason is that, even though the spinodal values of field are at a fraction of $1/\sqrt{3}$ of the distance to the bottom of the potential, this distance is very much larger than the initial field fluctuations. In consequence, the fluctuations in the field are now too small to create new lines and points of false vacuum of any substance and we can safely say that we have defects. To see this, we observe that, initially, the probability $p_t[\Phi]$, now independent of t, is expressable as

$$p[\Phi] = N\, e^{-\beta_0 H[\Phi]} \tag{8.165}$$

N is a normalisation factor and H permits the expansion in $\beta_0 m_0$ [15] in terms of S_0 of (5.110)

$$H[\Phi] = S_0[\Phi] - \frac{1}{24}\beta_0^2 \int d^D x \left(\frac{\delta S_0}{\delta \Phi}\right)^2 + O(\beta_0^4 m_0^4). \tag{8.166}$$

With calculational simplicity in mind we restrict ourselves to high initial temperatures ($\beta_0 m_0 \ll 1$) for which it is sufficient to retain only the first term, and take $\lambda_0 = 0$.

As with zeroes, fluctuations are defined with respect to some length scale L, say. The fields $\Phi_{L,a}$ coarse-grained to this length are defined in (5.117). The probability $p_L(\bar{\Phi})$ that $\Phi_L = \bar{\Phi}$ can be written as

$$p_L(\bar{\Phi}) = N\, \exp(-\bar{\Phi}^2/2(\Delta_L\Phi)^2) \tag{8.167}$$

where $(\Delta_L\Phi)^2)$ is the coarse-grained two-point function

$$\begin{aligned}(\Delta_L\Phi)^2 &= W_L(r;0) \simeq T_0 \int_{|\mathbf{k}|<L^{-1}} đ^D k \, \frac{1}{\mathbf{k}^2 + m_0^2}. \\ &= \int_{|\mathbf{k}|<L^{-1}} đ^D k \, \mathcal{C}(k)\end{aligned} \tag{8.168}$$

for $\mathcal{C}(k)$ of (5.116), compatible with (5.119). For $m_0 \approx M$ it follows[15] that, for $L = O(M^{-1})$ in $D = 3$ dimensions

$$(\Delta_L \Phi)^2 \simeq AMT_0 \tag{8.169}$$

where $A \simeq 10^{-1}$.

The condition that there be no overhang is thus

$$(\Delta_L \Phi)^2 \ll \phi_0^2 \tag{8.170}$$

There is no difficulty in satisfying (8.170) for small coupling, for which it just becomes $\lambda \ll 1$.

Although the strings of zeroes (or monopoles) are far apart they still do not yet quite provide the semiclassical network of defects that can be used as an input for numerical simulations since the fields have to relax dissipatively from their spinodal values to the true minima of the potential along the lines suggested earlier. Because they are too costly in energy to be produced by fluctuations, there will be some defect annihilation, but no creation, in this final phase of freezing in the defects and the $\bar{n}(t_{sp})$ calculated previously is an overestimate of the string density at the end of the transition. However, if it were the case for vortices that all the string was in loops it is difficult to see how infinite string could be created if all that happens is that string is removed from the system, although it cannot be precluded. Since that is the main question that we shall be addressing here we adopt the simplest assumption that, even if the density $\bar{n}(t_{sp})$ is an overestimate the distribution of strings (i.e. the fraction of strings in loops, the index for length distributions) is approximately unchanged by the final freezing in. That is, the one-scale scaling regime remains valid till freeze-in.

In that case the distribution of strings will then be as above for $\xi = \xi(t_{sp})$, while $\Delta\xi/\xi$ of (8.183) remains (albeit approximately) the variation in domain size over which field *phase* is correlated. Direct attempts to determine distributions from density correlation functions are difficult. For example, the simple analytic form of (7.149) enables us to calculate the density correlation function $C(r; t_{sp})$ for monopoles in the plane, and $A(r; t_{sp})$ and $B(r; t_{sp})$ for vortices in three dimensions, up to exponentially small terms in $Mt_\tau(t_{sp})$, as

$$C(r) = A(r) = \frac{2}{\pi^2 \xi_{sp}^4} \frac{e^{-2r^2/\xi_{sp}^2}}{(1 - e^{-2r^2/\xi_{sp}^2})^2} \left[(1 - e^{-2r^2/\xi_{sp}^2}) - 2\frac{r^2}{\xi_{sp}^2} \right] < 0, \tag{8.171}$$

and

$$B(r) = \frac{2}{\pi^2 \xi_{sp}^4} \frac{e^{-2r^2/\xi_{sp}^2}}{(1 - e^{-2r^2/\xi_{sp}^2})} > 0, \tag{8.172}$$

and we have dropped unneccessary time labels.

In units of $\bar{n}^2(t_{sp})$, the anticorrelation is large. In particular, a calculation of $C(r)$ or $A(r)$ for defect separation r gives

$$A(r) = \bar{n}^2(t_{sp}) \left[-1 + O\left(\frac{r^2}{\xi_{sp}^2} \right) \right] \tag{8.173}$$

when $r < \xi_{sp}$, as happens for the single mode case. Again, as in the single-mode case,

$$B(r) = \bar{n}^2(t_{sp})\Big[\Big(\frac{\xi_{sp}^2}{r^2}\Big) + O(1)\Big] > 0. \tag{8.174}$$

Although (8.172) is not wholly reliable for $r \gg \xi$, the suggestion that C or A and B fall off very fast is qualitatively correct, showing that ξ_{sp} indeed sets the scale at which strings see one another. The best we can say is that, taken together these suggest a significant amount of string in small loops.

We have more success in using the correlation functions to determine the variance in vortex winding number N_S through an open surface S in the 1-2 plane, which we take to be a disc of radius R. There is a complication in that, as can be seen from (2.32), ρ_3 counts zeroes weighted by the cosine of the angle with which they pierce S, whereas winding number counts zeroes in S without weighting. The problem is therefore essentially a two-dimensional problem. Using the results of (2.28) onwards, it is not difficult to see that, given the short range of the correlation functions, for the density of zeroes

$$(\Delta N)^2_S = O(\bar{n}^2\xi_{sp}^3 R) = O(R/\xi_{sp}). \tag{8.175}$$

Since $(\Delta N)_S$ is $(\Delta\alpha)_S/2\pi$, where $(\Delta\alpha)_S$ is the variance in the field phase around the perimeter ∂S of S, this means in turn that

$$(\Delta\alpha)^2_S = O(R/\xi(t_{sp})). \tag{8.176}$$

If this were equally true for nonrelativistic vortex systems, then $(\Delta\alpha)^2_S$ measures the variance in the supercurrent produced by the quench[7] and such behaviour is measurable, in principle. In practice we do not yet know how to construct the coefficient of R for non-relativistic fields, but see Refs.[20, 21].

Returning to the main problem of the length distributions of vortices in three dimensions, it has not yet proved possible to turn expressions for the C_{ij} (i.e.A and B) directly into statements about self-avoidance, fractal dimension, or whatever is required to understand the resulting string network. Fortunately, as a temporary expedient, we can work indirectly by adapting the numerical results of [6], based on Gaussian fields with the two-point function $W_n(r;t)$ of (4.104) that we considered earlier. For $n = 0$ this has the same long-wavelength behaviour as our dynamical $W(r;t)$ of (5.133) and (5.135). The results of [6] for $n = 0$ reproduce those of [8] on a cubic lattice, with its preponderence of open string. This leads us to believe that the simple dynamical model that we have proposed above (given its assumptions for the freezing in of domains) would also lead to a large fraction of open string.

Extending the simulation to general n for $W_n(r;t)$ of (4.104) shows[6] that, in order to decrease the amount of string in open string, it is necessary to *decrease* n. However, although the fraction of infinite string does diminish as n becomes negative, infinite string only seems to vanish in the pathological limit $n \to -3$, at which there is an infrared divergence. We see from (4.105) that the density of string vanishes in the same limit. As long as there is a nonzero density of string, then some of it is infinite.

The only way to increase the power in long wavelengths in our dynamical model is to change $\mathcal{C}(k)$ by imposing different initial conditions. Altering $\mathcal{C}(k)$ from $\mathcal{C}(k) = O(k^0)$ to $\mathcal{C}(k) = O(k^n)$ leads to a change in the power of the long wavelength modes from $n = 0$ to n, corresponding to a modification of $W_M(r;t)$ of (5.133) to

$$W_M(r;t) \propto \int_{|\mathbf{k}|<M} dk\, sinc(kr)\, k^{2+n}\, e^{2\Omega(k)t}. \tag{8.177}$$

The effect of changing n can be seen qualititively by expanding $\exp\{\Omega(k)t\}$ as in (5.134). $W_M(r;t)$ of (8.177) can then be evaluated explicitly, giving $f^{(n)}(r;t)$ as the confluent hypergeometric function

$$\begin{aligned} f^{(n)}(r;t) &\approx \exp\{-r^2M/4t\}\, {}_1F_1\Big(\frac{-n}{2}; \frac{3}{2}; \frac{r^2M}{4t}\Big) \\ &= {}_1F_1\Big(\frac{n+3}{2}; \frac{3}{2}; -\frac{r^2M}{4t}\Big) \end{aligned} \tag{8.178}$$

on dropping the upper bound in the integral.

The densities obtained from the more complicated (8.178) are *exactly* those of the more simple $W_n(r;t)$ given in (4.105), if we identify k_0^2 as $M/2t$. Further, for $n > -2$ the integrand of $W(r;t)$ in (8.177) shows a peak whose width increases as

$$\Delta k \propto \frac{1}{\sqrt{n+2}}, \tag{8.179}$$

implying a variation in domain size that also increases as $n \to -2$, just as for the simpler case.

Since ${}_1F_1(0; 3/2; r^2M/4t) = 1$ identically, $n = 0$ is a special case in which we reproduce the Gaussian of (5.136). Although this seems at variance with the behaviour given in (4.108) for the simply cut-off power distribution, the exponential behaviour was never supposed to be valid for very large r. Despite that, for $n < 0$ the similarity is good, with the *same* power-law falloff for $n = -1$ and $n = -2$, leading to anticorrelations over increasingly longer ranges.

Thus, if we can force $\mathcal{C}(k)$ to have negative n, the numerical simulations of [6] suggest that there will be more string in loops, and less in infinite string. The reason why we have $n = 0$ white noise in (5.133) is that, when quenching from *well above* the transition, $\mathcal{C}(k) \approx T_0/m_0^2$ is constant for the dominant integration region. In attempting to alter $\mathcal{C}(k)$ we expect that, if we were to begin closer to the transition where the initial fluctuations are larger, we can make n negative. On beginning a quench from thermal equilibrium, the most extreme case is one in which we start from a temperature so close to the transition that the effective mass m_0 of (5.112) for $t < 0$ is approximately zero. Let us first suppose that we could set it *identically* zero. Then

$$\mathcal{C}(k) = \frac{T_0}{k^2}. \tag{8.180}$$

rather than (5.127). The k^{-2} behaviour cancels the k^2 behaviour coming from the radial momentum integration and the power in long wavelengths is increased from $n = 0$ to $n = -2$ in (8.177). We believe that this is the most negative that n can become. Even if we had performed a more honest calculation of (7.153) there is no way for the interactions to introduce the singular infrared behaviour of $n = -3$ that is necessary to prohibit the production of strings. The simulations of [6] suggest an approximate halving of the fraction of open string for $n = -2$ from its original 80% (on a cubic lattice). These numbers are not to be taken too seriously since, as we commented earlier, they depend on the type of lattice[9] (although the presence of open string does not).

As we had noted above, for such a $\mathcal{C}(k)$ there is no peaking in the integrand around any wavenumber, and so no length that characterises a domain size, although the string density is nonzero. From (8.168) we see that the same $\mathcal{C}(k)$ of (8.180) also determines the initial field fluctuations. However, the effect there is small, contrary to naive expectation, provided we coarse-grain to the scale appropriate to defects, $k < M$. This is the relevant scale, rather than the thermal wavelength β_0. On coarse-graining to $k < M$ the field fluctuations, given by (8.168), still satisfy $(\Delta_L\Phi)^2 \simeq AMT_0$, and the initial fluctuations will not populate the ground states for weak coupling.

Our ability to make n negative survives more realistic initial states. The situation is more complicated in that the Gaussian approximation based on (5.110) breaks down[15] at temperatures T closer to T_c than

$$\left|1 - \frac{T^2}{T_c^2}\right| = O(\lambda). \tag{8.181}$$

Thus the minimum value of m_0^2 at which our approximations have any hope is $m_0^2 = O(\lambda M^2)$. However, this is small enough for (8.180) to be a good approximation, provided

$$\lambda \ln(1/\lambda) \ll 1, \tag{8.182}$$

as follows on implementing the approximation (5.134). In practice quenching from so close is artificial, and we should consider $n = -2$ as an unattainable lower bound.

However, this is not the end of the story. The regular lattices of [8] and [6] are artefacts of the calculational scheme and may, of themselves, predict more infinite string than is present if they try to emulate a continuous transition. As we saw earlier, we do not have a regular domain structure in our model but have domains with a large variance

$$\frac{\Delta\xi}{\xi} \approx \frac{\Delta k}{k_c} = O(1) \tag{8.183}$$

independent of time. Whatever the details, the dispersion in ξ is large.

Although a regular lattice does not imply exactly regular domains (which we understand as characterising the spaces between the defects) for high densities of the kind we have here they do imply a greater regularity than we have. In numerical simulations of string networks the inclusion of variance in the 'lattice' cell size shows[5] that, the greater

the variance, the more string is in small loops. This can be understood in the following way. The strings generated by phase separation on regular lattices are known to behave like random walks in $D = 3$ dimensions to a very good approximation[22]. On a rectangular lattice the fraction of string in loops is only about 20%[8]. Increasing the variance of the domain size increases the 'target area' that a string must hit for a section of string to be deemed closed. The probability of finding loops therefore increases. The variance of (8.183), if it could be carried over to [5] as it stands, suggests much more string (*e.g.* twice or more) in small loops. However, it is not easy to marry the somewhat different distributions of this simulation to ours since ours is one of overlapping domains (because of the continuous transition) rather than a domain 'bubble' picture more appropriate to first-order transitions that is more easily accomodated by [5].

The end result is that there always seems to be infinite string from a continuous transition, even if less than we thought, although its confirmation will require numerical analysis along the lines of [6]. This will be performed elsewhere when we have a better understanding of how back-reaction stops domain growth[16] than the simple approximation presented here.

9 Conclusions

In this paper we have shown how global O(2) vortices (and monopoles) appear in a relativistic theory, at a quench from the ordered to disordered state, as a consequence of the growth of unstable Gaussian long wavelength fluctuations.

Assuming a simple freezing of defects the resulting string (and monopole) configurations scale as a function of the correlation length $\xi = O(\sqrt{t_\tau})$, for some effective time t_τ, at a fraction of a defect/correlation area. The label τ characterises the time it takes the defects to freeze in. The slower the quench (for weak coupling), the larger t_τ, and hence the lower the defect density. For quenches from well above the transition the results are insensitive to initial conditions, which have been assumed here to be thermal equilibrium, but possibly the result is more general. This is compatible with the Kibble mechanism for vortex production due to domain formation upon phase separation by white noise fluctuations (and similarly for monopole production). For a weak coupling theory the domain cross-sections are significantly larger than a vortex (or monopole) cross-section at the largest times for which the approximations are valid. Moreover, there is a large variance in their size ξ, with $\Delta\xi/\xi = O(1)$. Numerical simulations that relate loop distribution to the variance in domain size of the long wavelength modes suggest that there is likely to be more string in small loops than we might have anticipated on the basis of simulations on regular lattices, although the match with the simulations is not exact.

As we quench from closer to the transition and effectively reduce n we expect that the fraction of string in loops increases (as the density of string decreases), but even then it is never possible to enhance long wavelength fluctuations to an extent that there is no infinite string after a continuous transition.

We stress the crudity of some of the approximations that we have made. There

is little difficulty, in principle and in practice, in doing somewhat better. However, without any specific choice of initial conditions and mass evolution $m(t)$ to guide us, and without better-matching numerical simulations, generic results at this level are sufficient. Improvements, motivated by particular cosmological models, are being considered.

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